Probability in Computing

CS 237

Reminders

- HW 8 is due Thursday

Reading

- LLM 19.4-19.5
- LLM 20.2, 20.3
- P 3.2.2, 3.2.3

LECTURE 17

Last time

- Expectation and infinite sums
- Linearity of expectation

Today

- Expectation of continuous random variables
- Product of independent random variables
- Law of the Unconscious Statistician (LOTUS)
- Conditional expectation
- Law of Total Expectation

Top Hat question (Join Code: 033357)

- Example: permutations
  - 3 students exchange their hats, so that everybody gets a random hat
  - Let X be the number of students that got their own hats.
  - E.g., if students 1,2,3 got hats 2,1,3 then X=1.
- PMF of X:

$$\Pr(X = 0) = \frac{1}{3}, \Pr(X = 1) = \frac{1}{2}, \Pr(X = 3) = \frac{1}{6}.$$

- What's the expectation of X?

A. 3/4

B. 1

C. 3/2

D. None of the above.

Example: Random Hats

- Example: permutations
  - n students exchange their hats, so that everybody gets a random hat
  - Let X be the number of students that got their own hats.
- What's the expectation of X for general n?

Expectation for Continuous RVs

- The expectation of a continuous random variable is defined as in the discrete case, but we replace the sum with an integral
- Let X be a random variable with PMF f_X (discrete case) or PDF f_X (continuous case)

discrete:

$$\mathbb{E}(X) = \sum_{x \in \mathbb{R}} x \cdot f_X(x)$$

continuous:

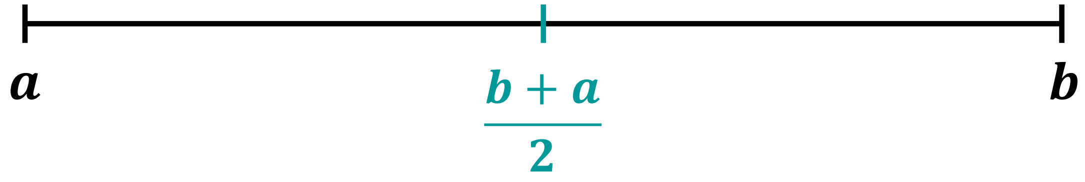
$$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx$$

Example: Uniform Random Variable

- Suppose X is chosen uniformly at random from the interval $[a, b]$
- The PDF is $f_X(x) = \frac{1}{b-a}$ for $x \in [a, b]$
- $$\mathbb{E}(X) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx = \int_a^b x \cdot \frac{1}{b-a} dx = \frac{1}{b-a} \int_a^b x\, dx$$

$$= \frac{1}{b-a} \left. \frac{x^2}{2} \right|_a^b = \frac{1}{b-a} \frac{b^2 - a^2}{2} = \frac{b+a}{2}$$

- Note: $\mathbb{E}(X)$ is the midpoint of the interval $[a, b]$



Example: Darts

- Suppose we throw a dart at a circular board of radius r
- Let D be the distance from the center of the board to the hitting point of the dart
- The PDF is $f_X(x) = \frac{2x}{r^2}$ for $x \in [0, r]$
- $\mathbb{E}(D) = \int_{-\infty}^{\infty} x \cdot f_X(x)\, dx \quad = \int_0^r x \cdot \frac{2x}{r^2} dx \quad = \frac{2}{r^2} \int_0^r x^2\, dx$

$$= \frac{2}{r^2} \left. \frac{x^3}{3} \right|_0^r \quad = \frac{2}{r^2} \frac{r^3}{3} \quad = \frac{2r}{3}$$

Functions of Random Variables

- A random variable can be a function of another random variable
- Examples: $Y = 3X + 1; Y = X^2; Y = |X|$
- If Y is a function of X then PMD/PDF of Y is related to the PDF of X

- Law of the Unconscious Statistician (LOTUS): If $Y = g(X)$ then

discrete: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \sum_{x \in \mathbb{R}} g(x) \cdot f_X(x)$$

continuous: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

LOTUS: Examples

- **Law of the Unconscious Statistician (LOTUS):** If $Y = g(X)$ then

discrete: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \sum_{x \in \mathbb{R}} g(x) \cdot f_X(x)$$

continuous: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

- **Example 1:** Let X be uniform over $\{-4, \dots, 4\}$. Then PMF of X is

$$f_X(x) = \begin{cases} \frac{1}{9} & \text{if } x \in \{-4, \dots, 4\} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X^2) = \sum_{x \in \mathbb{R}} x^2 \cdot f_X(x) = \sum_{x \in \{-4,\dots,4\}} x^2 \cdot \frac{1}{9} = \frac{2}{9}(1^2 + 2^2 + 3^2 + 4^2)$$

Without LOTUS: Examples

- Example 1: Let X be uniform over $\{-4, \dots, 4\}$. Then PMF of X is

$$f_X(x) = \begin{cases} \frac{1}{9} & \text{if } x \in \{-4, \dots, 4\} \\ 0 & \text{otherwise} \end{cases}$$

$$\text{Range}(X^2) = \{0,1,4,9,16\} = \{0,1^2,2^2,3^2,4^2\}$$

$$f_{X^2}(k) = \begin{cases} \frac{2}{9} & \text{if } k \in \{1,4,9,16\} \\ \frac{1}{9} & \text{if } k \in \{0\} \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X^2) = \sum_{k \in \{1^2,2^2,3^2,4^2\}} k \cdot \frac{2}{9} = \frac{2}{9}(1^2 + 2^2 + 3^2 + 4^2)$$

LOTUS: Examples

- Law of the Unconscious Statistician (LOTUS): If $Y = g(X)$ then

discrete: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \sum_{x\in\mathbb{R}} g(x) \cdot f_X(x)$$

continuous: $$\mathbb{E}(Y) = \mathbb{E}(g(X)) = \int_{-\infty}^{\infty} g(x) \cdot f_X(x)\, dx$$

- Example 2: Let X be uniform over [0,1]. Then PDF of X is

$$f_X(x) = \begin{cases} 1 & \text{if } x \in [0,1] \\ 0 & \text{otherwise} \end{cases}$$

$$\mathbb{E}(X^2) = \int_{-\infty}^{\infty} x^2 \cdot f_X(x)\, dx = \int_0^1 x^2\, dx = \left.\frac{x^3}{3}\right|_0^1 = \frac{1}{3}$$

Product of independent RVs

- Theorem. For any two **independent** random variables X and Y on the same probability space,

$$\mathbb{E}(X \cdot Y) = \mathbb{E}(X) \cdot \mathbb{E}(Y)$$

- Note. The equality does not hold, in general, for dependent random variables.

Example. We toss two coins.

Let X= number of HEADS, Y= number of TAILS.

Calculate $\mathbb{E}(X)$, $\mathbb{E}(Y)$ and $\mathbb{E}(XY)$.

Product of independent RVs

- Find the ranges and probabilities

Outcomes	X	Y	$X \cdot Y$	**Pr**
{H,H}	2	0	0	1/4
{H,T}	1	1	1	1/4
{T,H}	1	1	1	1/4
{T,T}	0	2	0	1/4

Product of independent RVs

- Find the ranges and probabilities

Outcomes	X	$2 - X$	$X \cdot (2 - X)$	**Pr**
{H,H}	2	0	0	1/4
{H,T}	1	1	1	1/4
{T,H}	1	1	1	1/4
{T,T}	0	2	0	1/4

Product of independent RVs

- Find the ranges and probabilities

X	Y	$X \cdot Y$	**Pr**
2	0	0	1/4
1	1	1	1/4
1	1	1	1/4
0	2	0	1/4

$$\mathbb{E}(X) = \sum_{x \in R_X} x \cdot Pr(X = x)$$

$$\mathbb{E}(Y) = \sum_{y \in R_Y} x \cdot Pr(Y = y)$$

$$\mathbb{E}(X \cdot Y) = \sum_{y \in R_X} \sum_{y \in R_Y} x \cdot y \cdot \Pr(X = x, Y = y)$$

Product of independent RVs

- Find the ranges and probabilities

X	Y	$X \cdot Y$	**Pr**
2	0	0	1/4
1	1	1	1/4
1	1	1	1/4
0	2	0	1/4

$$\mathbb{E}(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$$\mathbb{E}(Y) = \sum_{y \in R_Y} x \cdot Pr(Y = y)$$

$$\mathbb{E}(X \cdot Y) = \sum_{y \in R_X} \sum_{y \in R_Y} x \cdot y \cdot \Pr(X = x, Y = y)$$

Product of independent RVs

- Find the ranges and probabilities

X	Y	$X \cdot Y$	Pr
2	0	0	1/4
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$$\mathbb{E}(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$$\mathbb{E}(Y) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$$\mathbb{E}(X \cdot Y) = \sum_{y \in R_X} \sum_{y \in R_Y} x \cdot y \cdot \Pr(X = x, Y = y)$$

Product of independent RVs

- Find the ranges and probabilities

X	Y	$X \cdot Y$	**Pr**
2	0	0	1/4
1	1	1	1/4
1	1	1	1/4
0	2	0	1/4

$$\mathbb{E}(X) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$$\mathbb{E}(Y) = 0 \cdot \frac{1}{4} + 1 \cdot \frac{1}{2} + 2 \cdot \frac{1}{4} = 1$$

$$\mathbb{E}(X \cdot Y) = 2 \cdot 0 \cdot \frac{1}{4} + 1 \cdot 1 \cdot \frac{1}{2} + 0 \cdot 2 \cdot \frac{1}{4} = \frac{1}{2}$$

In this example, the equality does not hold

☹

Top Hat question (Join Code: 033357)

For arbitrary random variables X and Y, by linearity of expectation:

A. $\mathbb{E}(X+Y)\ = \mathbb{E}(X) + \mathbb{E}(Y)$

B. $\mathbb{E}(aX+bY)\ = a\mathbb{E}(X) + b\mathbb{E}(Y)$ for all $a, b \in \mathbb{R}$.

C. $\mathbb{E}(XY) = \mathbb{E}(X)\ \mathbb{E}(Y)$

D. Both A and B are correct.

E. A,B and C are correct.

Conditional Expectation

- Definition: The conditional expectation of a discrete random variable X given an event A is

$$\mathbb{E}(X \mid A) = \sum_{x \in R_X} x \cdot \Pr(X = x \mid A)$$

$$= \sum_{x \in R_X} x \cdot \frac{\Pr(X = x \ \cap \ A)}{\Pr(A)}$$

Conditional Expectation

- We can also have a condition involving another random variable:

$$\mathbb{E}(X \mid Y = y) = \sum_{x \in R_X} x \cdot \Pr(X = x \mid Y = y)$$

$$= \sum_{x \in R_X} x \cdot \frac{\Pr(X = x \cap Y = y)}{\Pr(Y = y)}$$

Conditional Expectation

- Example:

$$\mathbb{E}\,(\,X\mid A\,) = \sum_{x\in R} x\cdot Pr(X = x\mid A)$$

$$\mathbb{E}\,(\,X\mid X\geq a\,) = \sum_{x\in R} x\cdot Pr(X = x\mid X\geq a)$$

X = number obtained when rolling a die

A = “number rolled is at least 4”

$$\{\,1,\,2,\,3,\,\underbrace{4,\,5,\,6}\,\}$$

$$\mathbb{E}(X|A) = \mathbb{E}(X|X\geq 4) = \quad 5$$

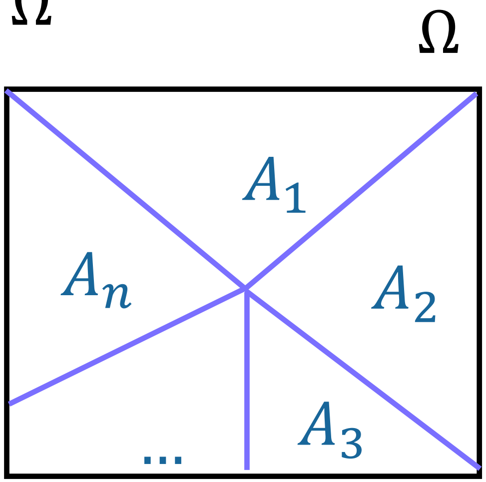
Law of Total Expectation

- Using the Law of Total Probability, we can show:

$\mathbb{E}(X) = \mathbb{E}(X|A_1) \cdot \Pr(A_1) + \mathbb{E}(X|A_2) \cdot \Pr(A_2) + \ldots + \mathbb{E}(X|A_n) \cdot \Pr(A_n)$

X is a random variable

A1, A2, …, An is a partition of the sample space Ω



Law of Total Expectation

- Using the Law of Total Probability, we can show:

$$\mathbb{E}(\mathrm{X}) = \mathbb{E}(\mathrm{X}|\mathrm{A}_1)\cdot\Pr(\mathrm{A}_1) + \mathbb{E}(\mathrm{X}|\mathrm{A}_2)\cdot\Pr(\mathrm{A}_2) + \ldots + \mathbb{E}(\mathrm{X}|\mathrm{A}_\mathrm{n})\cdot\Pr(\mathrm{A}_\mathrm{n})$$

X = number obtained when rolling a die

A = number rolled is at least 5

B = number rolled is at most 4

$$\{\,\underbrace{1,\ 2,\ 3,\ 4}_{\mathrm{B}},\ \underbrace{5,\ 6}_{\mathrm{A}}\,\}$$

$$\mathbb{E}(\mathrm{X}) = \mathbb{E}(\mathrm{X}|A)\cdot\Pr(\mathrm{A}) + \mathbb{E}(\mathrm{X}|B)\cdot\Pr(\mathrm{B})$$

$$= \mathbb{E}(\mathrm{X}|A)\cdot\frac{1}{3} + \mathbb{E}(\mathrm{X}|B)\cdot\frac{2}{3} = \frac{5.5}{3} + \frac{2\cdot 2.5}{3} = \frac{10.5}{3} = 3.5$$

Example: More Dice

- Let X = number of dots showing on two rolled dice
- Let A = the first roll was ≥ 4

2nd

1st

	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Naive approach: Just assume the event is the whole sample space:

$$E(X \mid A) =$$

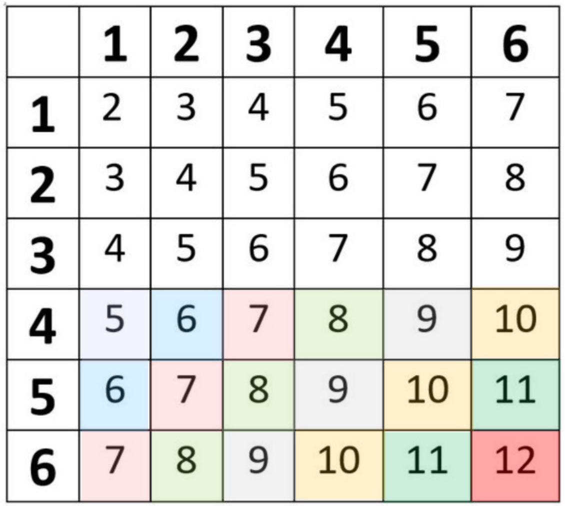
$$\frac{(5+\cdots+10)+(6+\cdots+11)+(7+\cdots+12)}{18}=\frac{153}{18}=8.5$$

Example: More Dice

- Let X = number of dots showing on two rolled dice
- Let A = the first roll was ≥ 4

2nd

1st



	1	2	3	4	5	6
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3	4	5	6	7	8	9
4	5	6	7	8	9	10
5	6	7	8	9	10	11
6	7	8	9	10	11	12

Using Conditional Expectation:

$$E(X \mid A) = \sum_{x \in \mathbf{R}} x \cdot \frac{\Pr(X = x \wedge A)}{\Pr(A)}$$

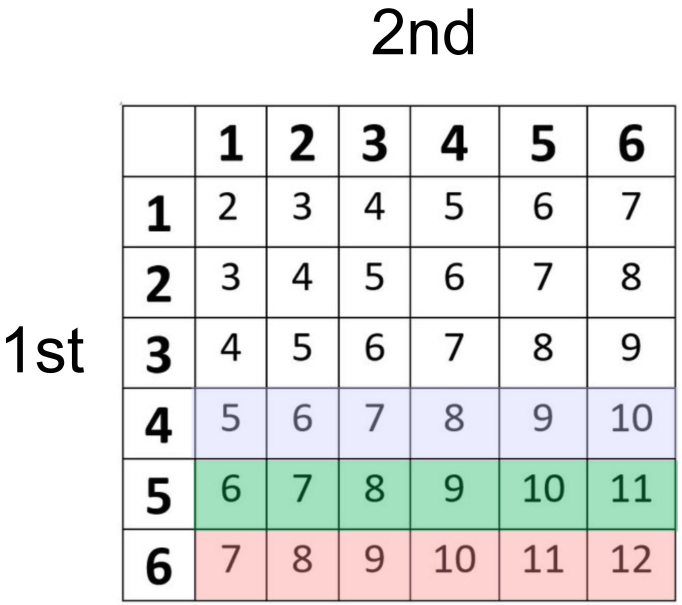
$$= \frac{5 \cdot \frac{1}{36} + 6 \cdot \frac{2}{36} + (7+8+9+10) \cdot \frac{3}{36} + 11 \cdot \frac{2}{36} + 12 \cdot \frac{1}{36}}{\frac{1}{2}} = 8.5$$

Example: More Dice

- Let X = number of dots showing on two rolled dice
- Let A = the first roll was ≥ 4

Using the Law of Total Expectation on a partition of an event:

$$E(X \mid A) = \frac{7.5 \cdot \frac{1}{6} + 8.5 \cdot \frac{1}{6} + 9.5 \cdot \frac{1}{6}}{\frac{1}{2}} = 8.5$$



2nd

1st

	1	**2**	**3**	**4**	**5**	**6**
1	2	3	4	5	6	7
2	3	4	5	6	7	8
3	4	5	6	7	8	9
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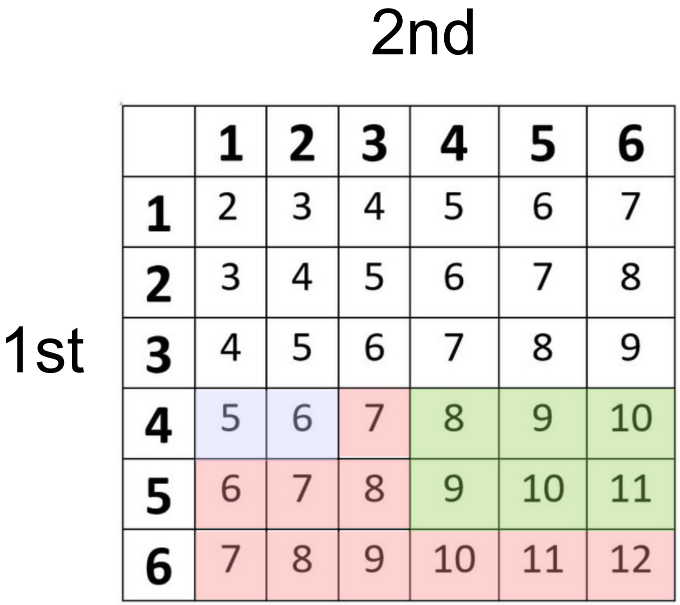
Example: More Dice

- Let X = number of dots showing on two rolled dice
- Let A = the first roll was ≥ 4

Using the Law of Total Expectation on a partition of an event:

$$E(X \mid A) = \frac{5.5 \cdot \frac{2}{36} + 8.5 \cdot \frac{10}{36} + 9.5 \cdot \frac{6}{36}}{\frac{1}{2}} = 8.5$$



2nd

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