Probability in Computing

LECTURE 17

CS 237

Last time

- Expectation of continuous random variables
- Product of independent random variables
- Law of the Unconscious Statistician (LOTUS)
- Conditional expectation
- Law of Total Expectation

Today

- Variance
- Discrete Distributions
  - Bernoulli
  - Uniform
  - Binomial
  - Geometric

Reminders

- HW 9 due Tonight

Reading

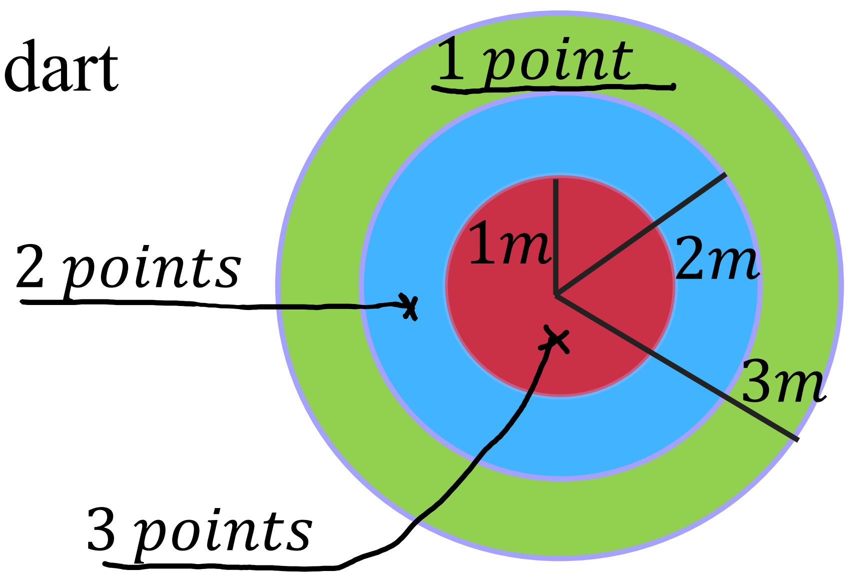
- LLM 20.2, 20.3; 19.3.1, 19.3.2,19.3.4

Playing with darts… again

- Dart board is divided into 3 regions, of radii 1, 2, and 3m
- If you throw a dart, you get 3, 2, or 1 points respectively
- Let X = Number of points you scored
- Let Y = the distance of your dart from center

What is your expected number of points given that the distance or your dart from the center is at most 2m?

$$\mathbb{E}\,(\,X \mid Y \leq 2\,) = \,?$$



Top Hat question (Join Code: 033357)

You roll two dice. Let X_1 be the value on the first die, X_2 be the value on the second die, and $X = X_1 + X_2$.

What is the value of $\mathbb{E}(X_1 | X = 5)$?

A. 0
B. 1/6
C. 2.5
D. 5
E. 6

Linearity of conditional expectation

- Theorem. For all random variables X and Y and all events A,
$$\mathbb{E}[X + Y \mid A] = \mathbb{E}[X \mid A] + \mathbb{E}[Y \mid A].$$
Also, for all $c \in \mathbb{R}$,
$$\mathbb{E}[cX \mid A] = c \cdot \mathbb{E}[X \mid A].$$

Exercise

You roll two dice. Let A be the event that you got no sixes.

Let X_1 be the value on the first die, X_2 be the value on the second die, and $X = X_1 + X_2$.

Calculate $\mathbb{E}[X \mid A]$

Expectation In Real Life

EXPECTATIONS IN THE REAL WORLD: INVESTING

- Investors are focused (rightly so) on *rates of return*
- Investments are often pitched by expected rates of return (ER), often based on historical estimates. For example:

Security X	1	2	3	4	5	6
Probability	0.05	0.10	0.30	0.15	0.25	0.35
Rate of Return	-7%	-3%	5%	8%	12%	15%

Security Y	1	2	3	4
Probability	0.20	0.40	0.45	0.25
Rate of Return	-15%	9%	10%	16%

- Here, ER[X] = 7.15%, ER[Y] = 9.1%, so Y is preferable.

Expectation In Real Life

EXPECTATIONS IN THE REAL WORLD: INVESTING

- But, expected rates of return are not the only consideration
  - "Past performance is no guarantee of future results"
- Also, some investments may be much riskier than others.

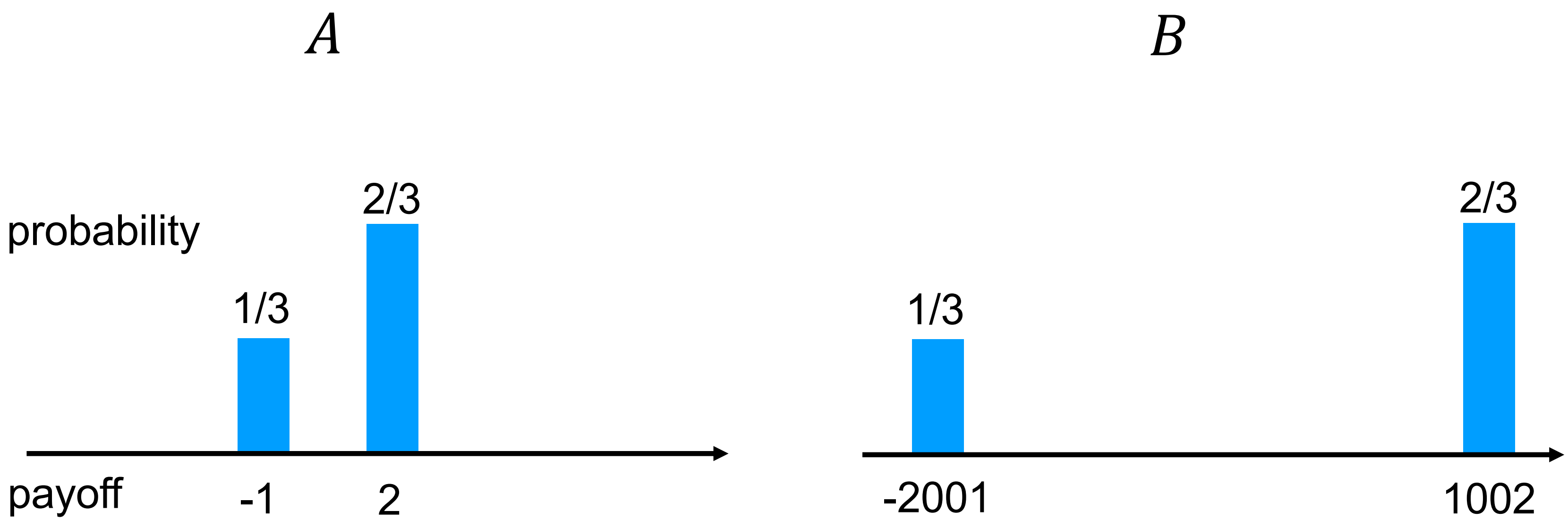
Security X	1	2	3	4	5	6
Probability	0.05	0.10	0.30	0.15	0.25	0.35
Rate of Return	-7%	-3%	5%	8%	12%	15%

Security Y	1	2	3	4
Probability	0.20	0.40	0.45	0.25
Rate of Return	-15%	9%	10%	16%

- Y has a 20% chance of a 15% loss. Not for everyone.
- Risk not captured by expectations and ER.

Variance: Let's play a game

Choose a game to play many times (say, every day for 1 year)

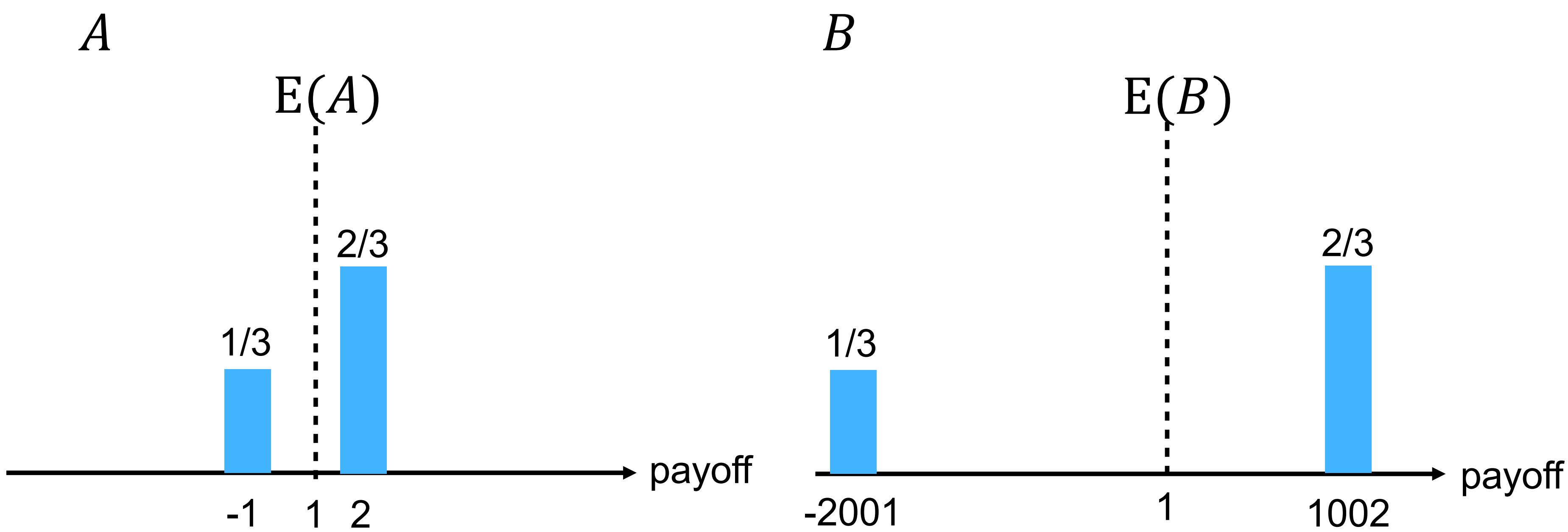


(A) I choose A (B) I choose B (C) I'm indifferent

(D) I'd rather not play either game

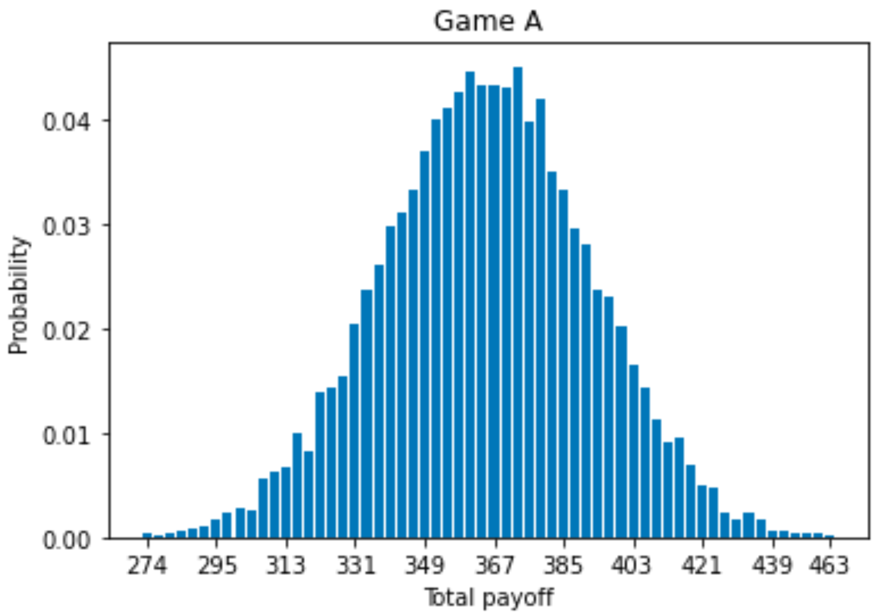
Variance: Let's play a game

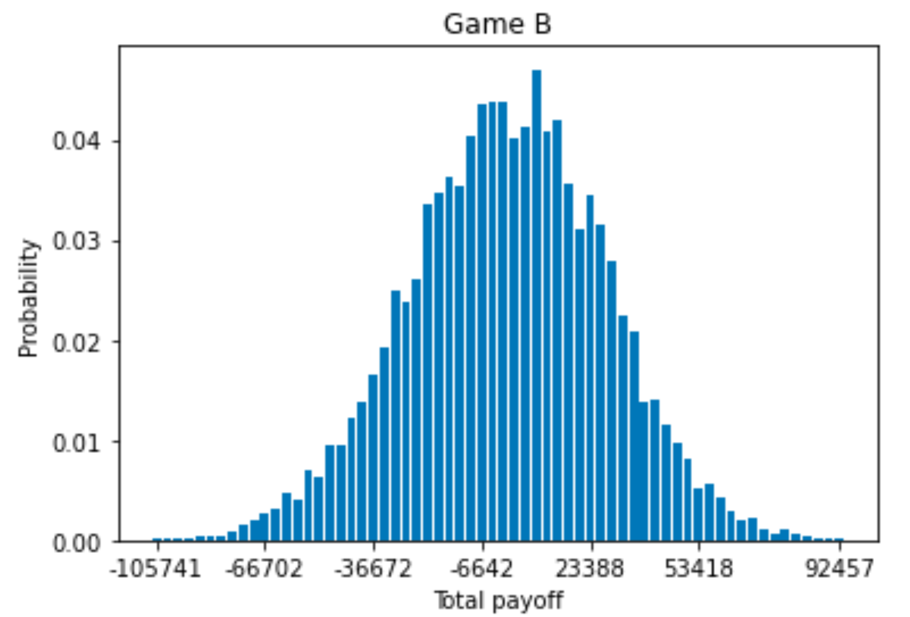
- Expected payoffs:



Variance: Let's play a game

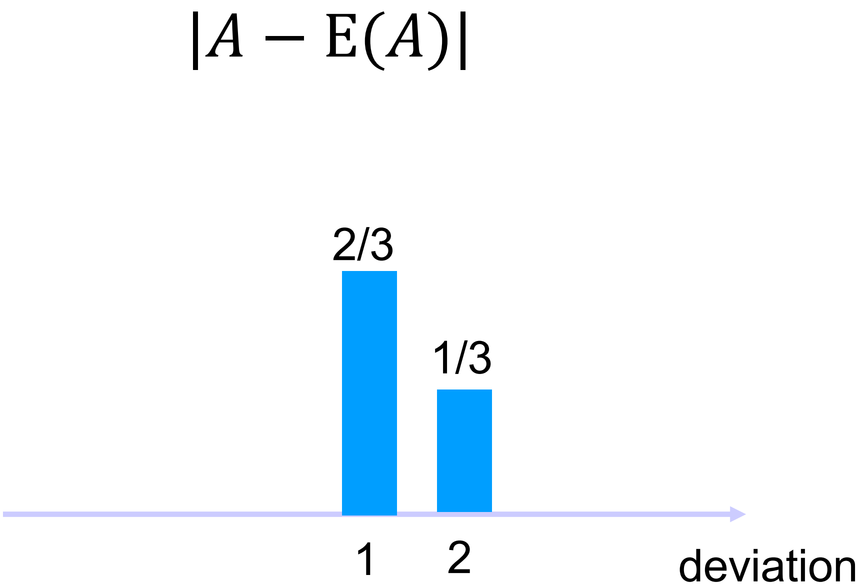
- Simulation:

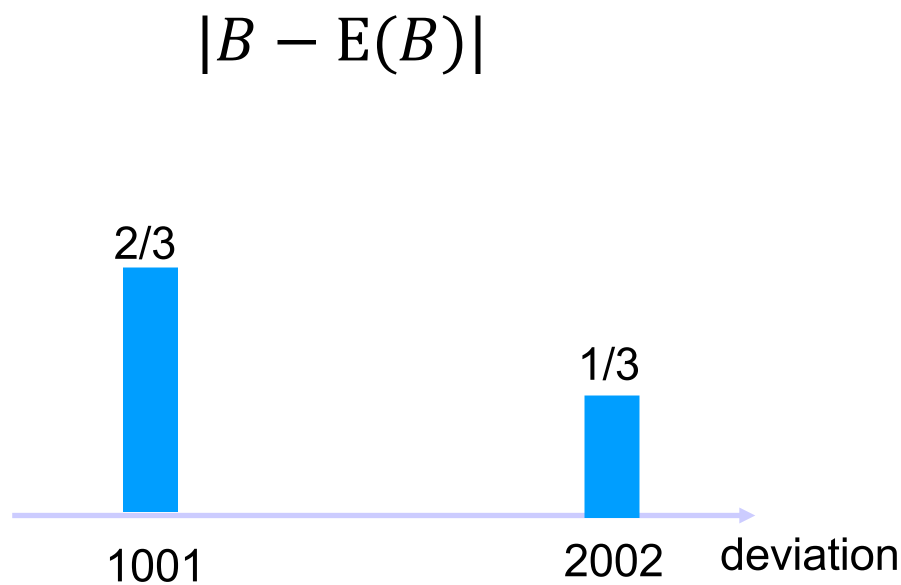




Variance: Let's play a game

- Deviation from the expectation:



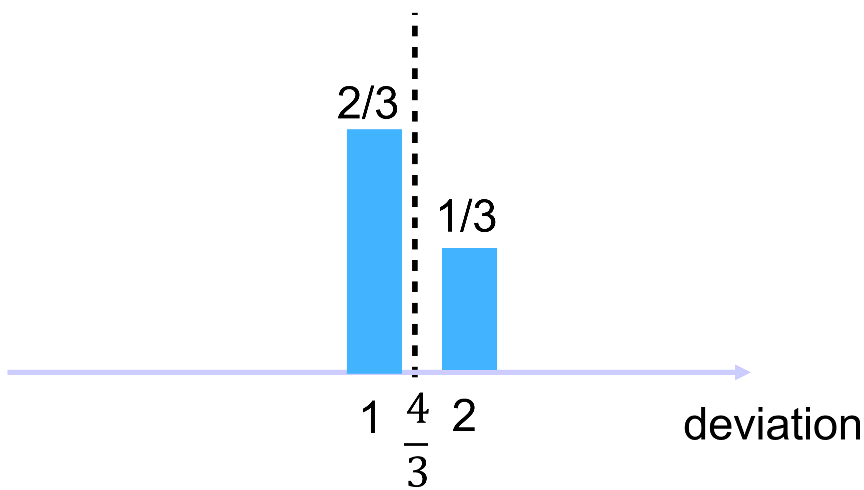


Variance: Let's play a game

- Deviation from the expectation:

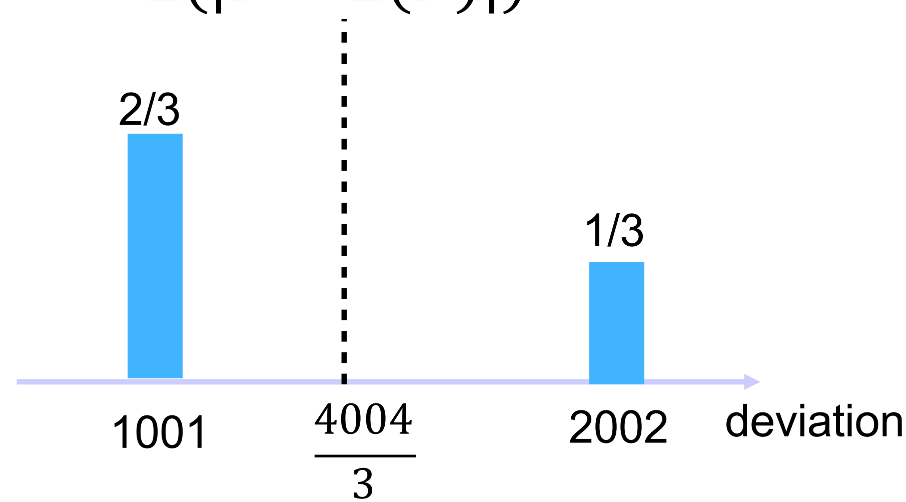
$$|A - \mathrm{E}(A)|$$

$$\mathrm{E}(|A - \mathrm{E}(A)|)$$



$$|B - \mathrm{E}(B)|$$

$$\mathrm{E}(|B - \mathrm{E}(B)|)$$

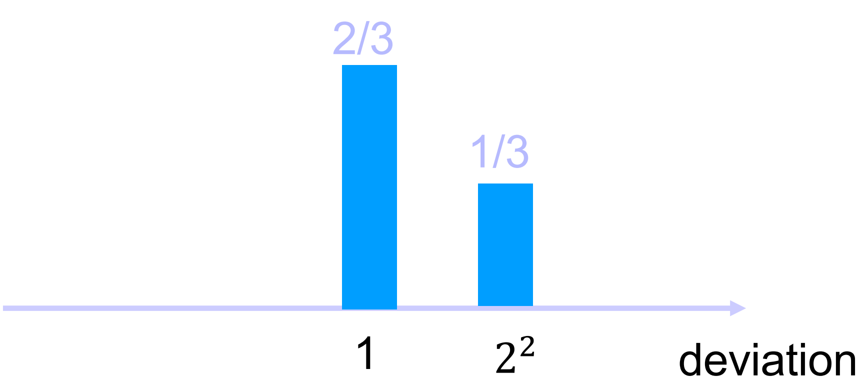


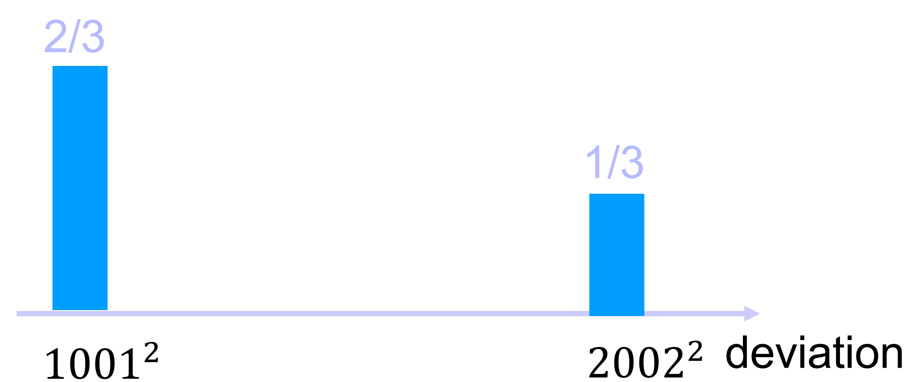
Variance: Let's play a game

- Deviation from the expectation:

$(A - \mathrm{E}(A))^2$

$(B - \mathrm{E}(B))^2$





The squared difference is nicer to work with than absolute value!

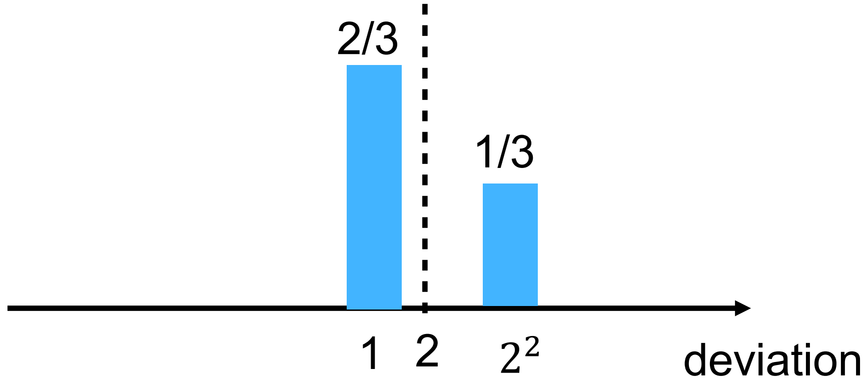
Variance: Let's play a game

- Deviation from the expectation:

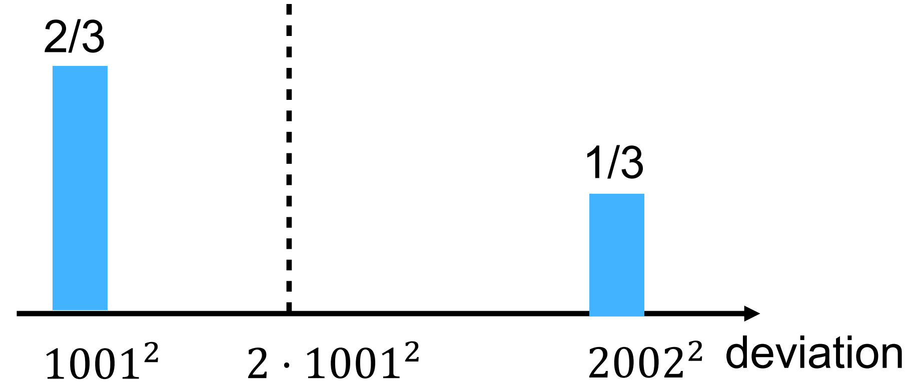
$(A - \mathrm{E}(A))^2$

$(B - \mathrm{E}(B))^2$

$\mathrm{E}((A - \mathrm{E}(A))^2)$



$\mathrm{E}((B - \mathrm{E}(B))^2)$



The value $\mathrm{E}((X - \mathrm{E}(X))^2)$ is called the variance of X

Variance Defined

- Definition: The variance of a random variable X is

$$\mathrm{Var}(X) = \mathrm{E}((X - \mathrm{E}(X))^2)$$

- As in the previous example, it is helpful to read the definition from the inside out:

$$X - \mathrm{E}(X) \to (X - \mathrm{E}(X))^2 \to \mathrm{E}((X - \mathrm{E}(X))^2)$$

Standard Deviation

- Definition: The standard deviation of a random variable X is
$$\sigma_X = \sqrt{\mathrm{Var}(X)} = \sqrt{\mathrm{E}((X - \mathrm{E}(X))^2)}.$$
- In the previous example, the unit of X was dollars but the unit of the variance is dollars squared.
- Since the units of variance are squared, we consider the square root of the variance.

Variance

- Theorem:

$$\mathrm{Var}(X) = \mathrm{E}(X^2) - (\mathrm{E}(X))^2$$

Example: Coin Toss

- Toss a coin that comes up Heads with probability p
- Let $X = \begin{cases} 1 & \text{if Heads} \\ 0 & \text{if Tails} \end{cases}$
- Since X is an indicator RV, we know $\mathrm{E}(X) = p$
- Var(X) = ?

Top Hat question (Join Code: 033357)

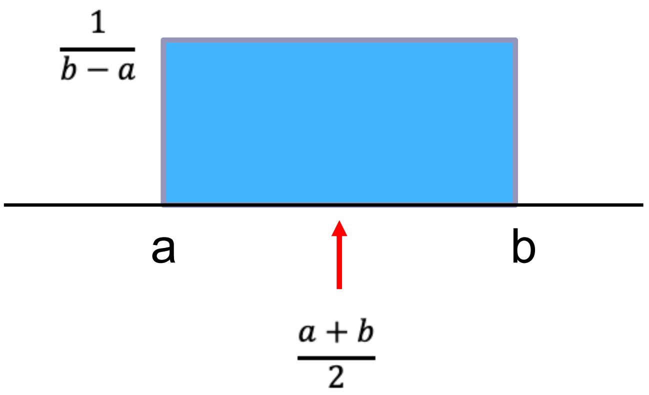
Let X be an indicator Random Variable with $\Pr(X = 1) = 1.0$

What is $\mathrm{Var}(X)$?

A. 0
B. 1
C. Impossible to know

Example: Continuous Uniform RV

- Let X be a continuous RV which chooses a value from [a,b] uniformly at random.
- The PDF is $f(x) = \begin{cases} \frac{1}{b-a} & \text{for } x \in [a, b] \\ 0 & \text{otherwise} \end{cases}$
- We have $\mathrm{E}(X) = \dfrac{a+b}{2}$



Var(X) =

Properties of variance

- By linearity of expectation, we have

$$\text{Ex}(aX + b) = a \cdot \text{Ex}(X) + b$$

- Does this hold for variance?

$$\text{Var}(aX + b) = ?$$

Properties of Variance

- Again, for expectation by linearity we have for any X and Y:

$$E(X+Y)=E(X)+E(Y)$$

- Does this hold for variance?

$$\mathrm{Var}(X+Y)\stackrel{?}{=}\mathrm{Var}(X)+\mathrm{Var}(Y)$$

Distributions: Motivation

- There are many random variables arising in different probability spaces with the same pdf/cdf
- Random variables with the same pdf/cdf behave very similarly and have formulae for the expected value, variance, etc.
- Can classify random variables into named distributions and study them in a unified way
- We will study the following discrete distributions:
  - Bernoulli
  - Uniform
  - Binomial

    Today
  - Geometric
  - Negative Binomial

    Next lecture
- Later on we will study continuous distributions:
  - Normal
  - Exponential

Bernoulli Distribution

Jacob Bernoulli
[1654 - 1705]

- Let X be an indicator random variable:

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

- X has the Bernoulli distribution with parameter p
- Notation: $X \sim \text{Bernoulli}(p)$
- Canonical experiment: Flip a coin whose probability of heads is p and count the number of heads showing.

Bernoulli Distribution

- X ~ Bernoulli(p):

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

- The PMF and CDF are:

$$f_p(x) = \begin{cases} 1-p & \text{if } x = 0 \\ p & \text{if } x = 1 \\ 0 & \text{otherwise} \end{cases} \qquad F_p(x) = \begin{cases} 0 & \text{if } x < 0 \\ 1-p & \text{if } 0 \le x < 1 \\ 1 & \text{if } x \ge 1 \end{cases}$$

Bernoulli Distribution

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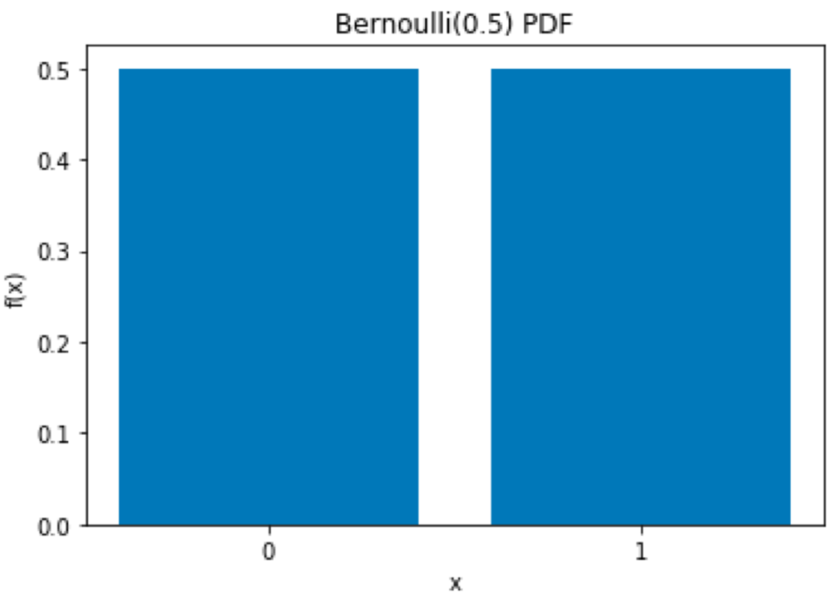

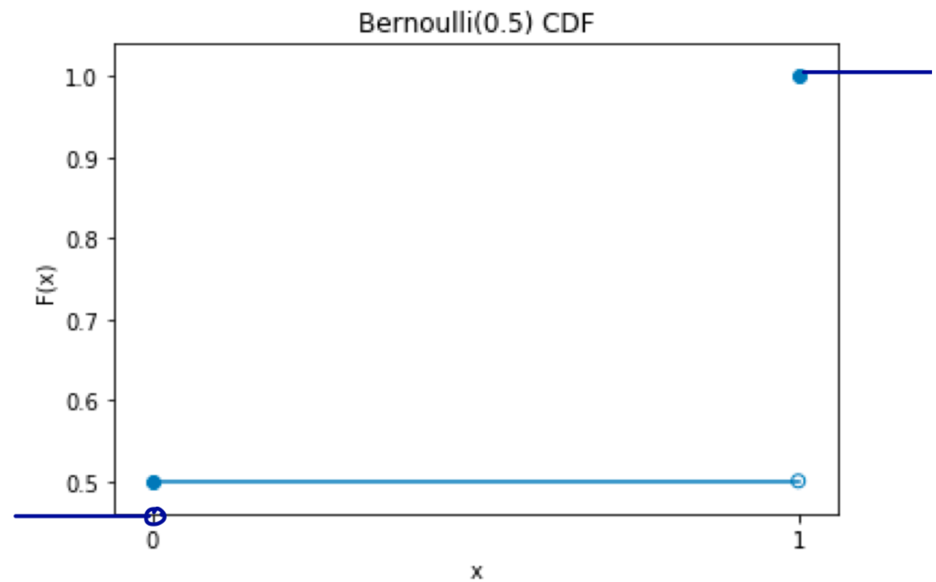

Bernoulli Distribution

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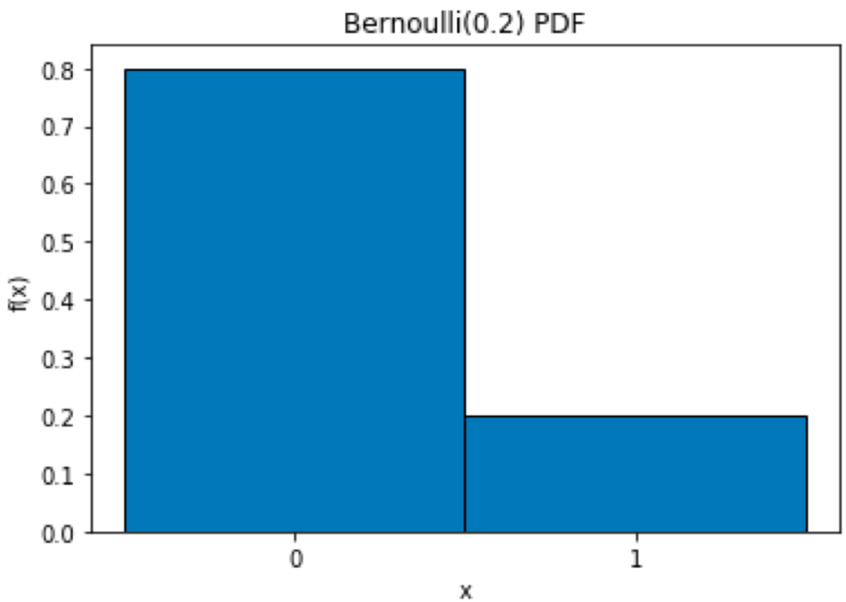
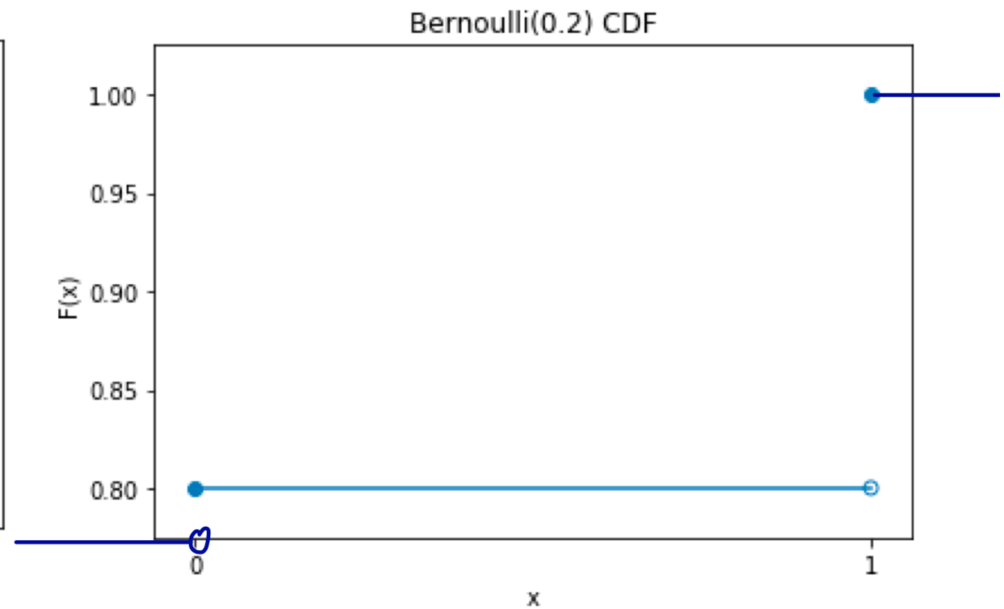




Bernoulli Distribution

- X ~ Bernoulli(p):

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1-p \end{cases}$$

- Models any scenario where there is a single trial that succeeds with probability p (X = 1 if success)
- Examples:
  - Will this coin show heads?
  - Will this test say I have Covid?
  - Am I pregnant?
  - Will I pass CS 237?

1 = "Success"

0 = "Failure"

Bernoulli Distribution

- $X \sim \text{Bernoulli}(p)$:

$$X = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } 1 - p \end{cases}$$

- Expectation: $\mathrm{E}(X) = p$
- Variance: $\mathrm{Var}(X) = p(1 - p)$

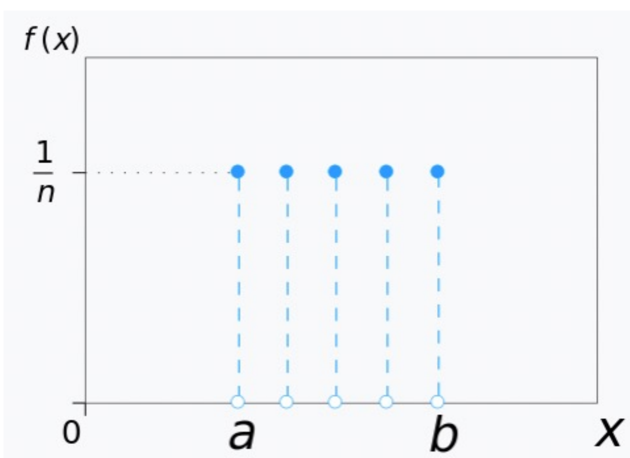
Uniform Discrete Distribution

- Let X be a random variable which uniformly chooses a number in [a, a+1, ..., b] and let N = length of sequence = b-a+1
- Notation: X ~ Uniform(a,b)
- PDF:

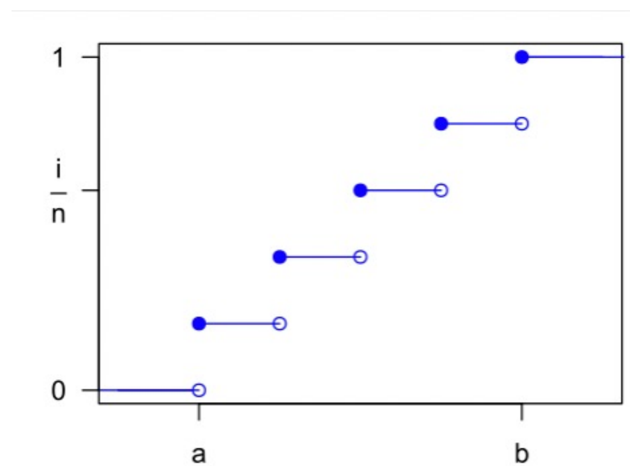
$$f(x) = \begin{cases} \frac{1}{N} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

- Canonical experiment: Roll an N-sided die labelled with numbers a through b

PMF



CDF



Uniform Discrete Distribution

- X ~ Uniform(a,b)

$$f(x) = \begin{cases} \frac{1}{N} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

- $E(X) =$

- Var(X) =

Uniform Discrete Distribution

- X ~ Uniform(a,b)

$$f(x) = \begin{cases} \frac{1}{N} & \text{for } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

for N = b – a + 1

- E(X) $= \dfrac{a+b}{2}$
- Var(X) $= \dfrac{N^2-1}{12}$