Probability in Computing

LECTURE 23

CS 237

Last time

- Applications of Markov and Chebyshev

Today

- Continuous Distributions:
    - **Uniform Distribution**
    - **Normal Distribution**

Reminders

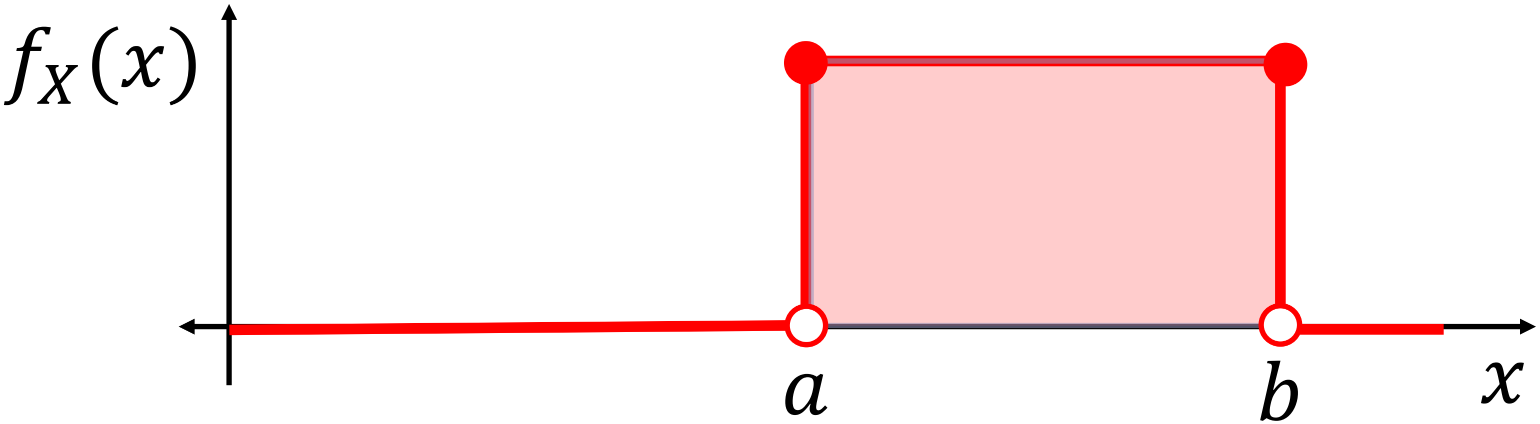
- HW 11 due Thursday

Reading

- P 4.2.1, 4.2.3

Continuous Uniform Distribution

- Also known as **Rectangular Distribution**



- $\Pr(a \le x \le b) = 1 = height \cdot widht$

PDF: Uniform Distribution

- A continuous random variable X is said to have a Uniform distribution over the interval $[a, b]$, shown as $X \sim Uniform(a, b)$, if its **PDF** is given by

$$f_X(x) = \begin{cases} \dfrac{1}{b-a} & a \leq x \leq b \\ 0 & x < a \text{ or } x > b \end{cases}$$

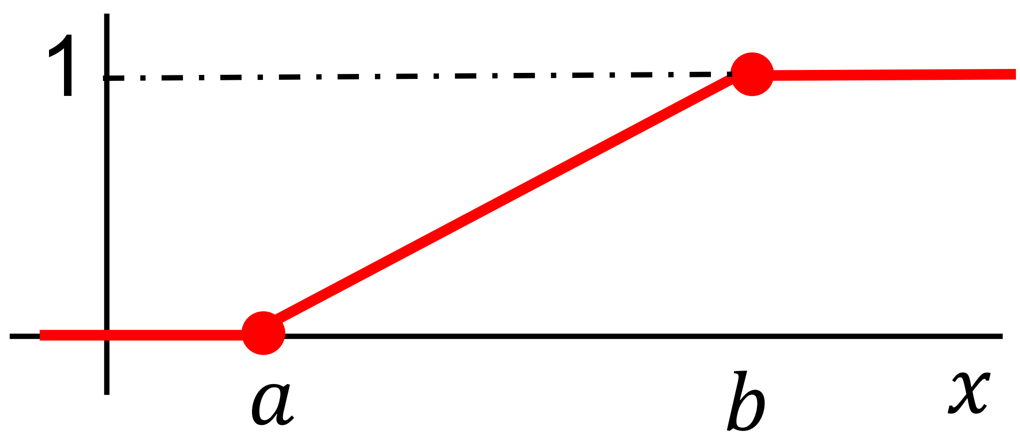
Top Hat question (Join Code: 033357)

- A student waits for the T between zero and 10 minutes, uniformly distributed. What is the probability that a student waits between 4 to 6 minutes?

CDF: Uniform Distribution

- By definition $F_X(x) = Pr(X \leq x)$. We have $F_X(x) = 0$ for $x < a$ and $F_X(x) = 1$ for $x > b$
- For $a \leq x \leq b$ we have

$$F_X(x) = Pr(X \leq x)$$
$$= Pr(X \in [a, x])$$
$$= \frac{x-a}{b-a}$$



CS 237 CDF: Uniform Distribution

- By definition $F_X(x) = Pr(X \le x)$. We have $F_X(x) = 0$ for $x < a$ and $F_X(x) = 1$ for $x > b$
- For $a \le x \le b$ we have
- $F_X(x) = \frac{x-a}{b-a}$
- To summarize

$$F_X(x) = \begin{cases} 0 & x < a \\ \dfrac{x-a}{b-a} & a \le x \le b \\ 1 & x > b \end{cases}$$

Top Hat question (Join Code: 033357)

A student waits for the T between zero and 10 minutes, uniformly distributed. What is the probability that the student waits at least 7 minutes?

Top Hat question (Join Code: 033357)

A student waits for the T between zero and 10 minutes, uniformly distributed. What is the expected waiting time?

Variance and Standard Deviation

A student waits for the T between zero and 10 minutes, uniformly distributed. What is the variance and standard deviation?

Conditional Uniform Probability

What is the probability that a student waits at most 8 minutes given that they waited at least 6 minutes?

Continuous Uniform Distribution

$$X \sim \text{Uniform}(a, b)$$

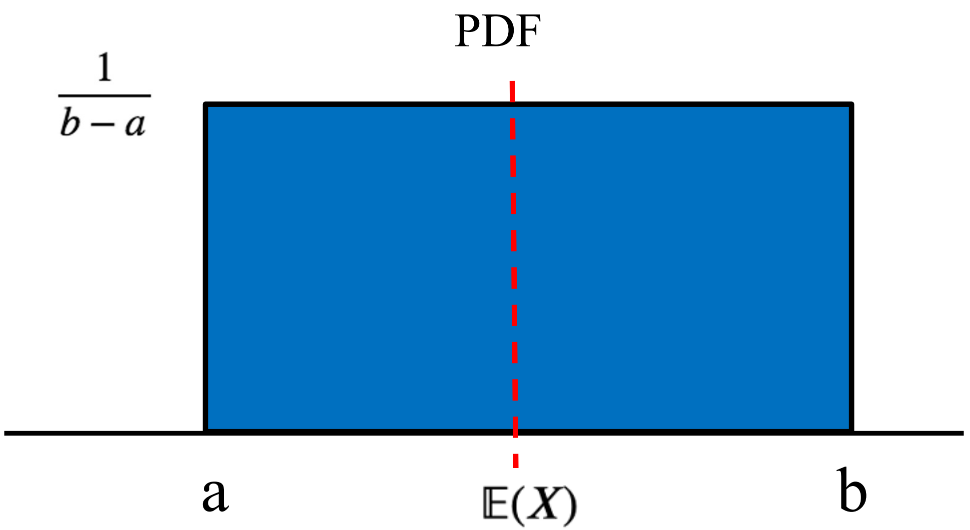
PDF: $$f_X(x) = \begin{cases} \dfrac{1}{b-a} & \text{if } a \le x \le b \\ 0 & \text{otherwise} \end{cases}$$

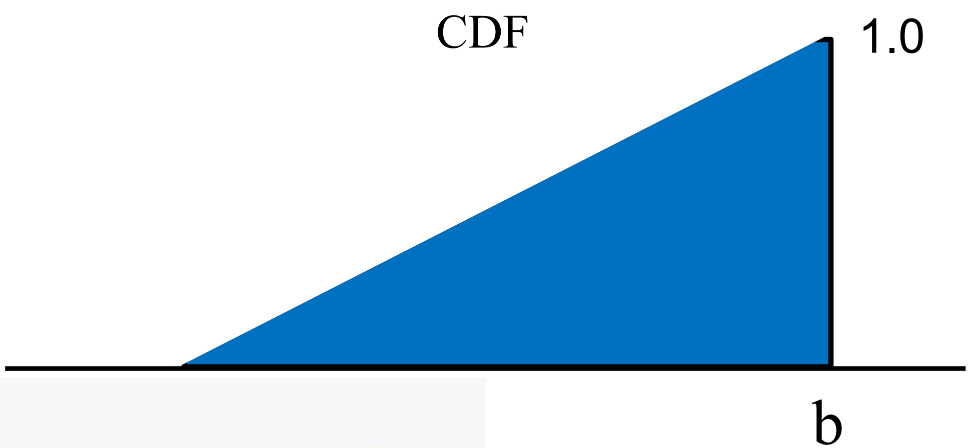
CDF: $$F_X(x) = \begin{cases} 0 & \text{if } x < a \\ \dfrac{x-a}{b-a} & \text{if } a \le x \le b \\ 1 & \text{if } x > b \end{cases}$$

$$\mathbb{E}(X) = \frac{a+b}{2}$$

$$Var(X) = \frac{(b-a)^2}{12}$$

$$\sigma_X = \frac{b-a}{\sqrt{12}}$$






```python
def X(a,b):
    return (b-a)*random() + a
```

Continuous Uniform Distribution

Reminder: The recommended way to calculate probabilities for continuous RVs is with intervals, usually using the CDF.

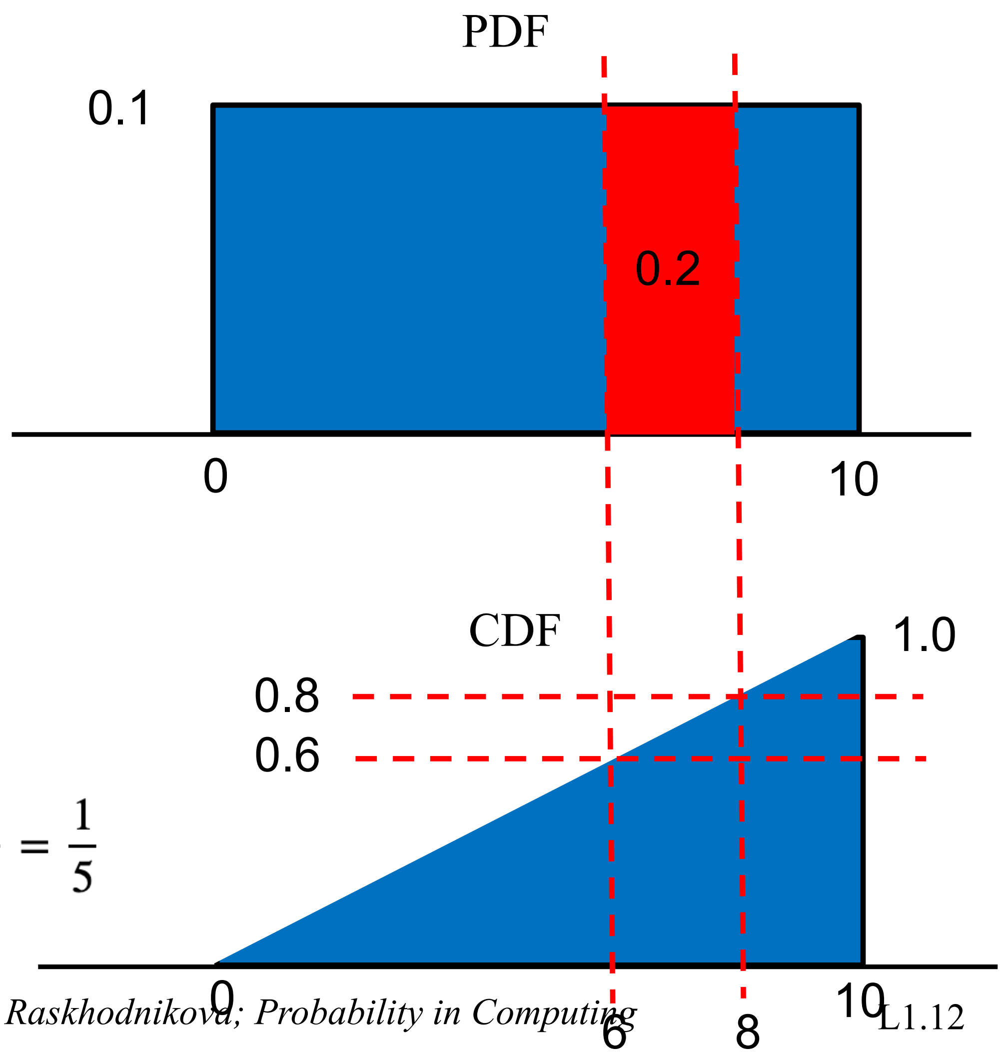
Example:

$$X \sim \text{Uniform}(0, 10)$$

$$f_X(x) = \begin{cases} \frac{1}{10} & 0 \leq x \leq 10 \\ 0 & \text{otherwise} \end{cases}$$

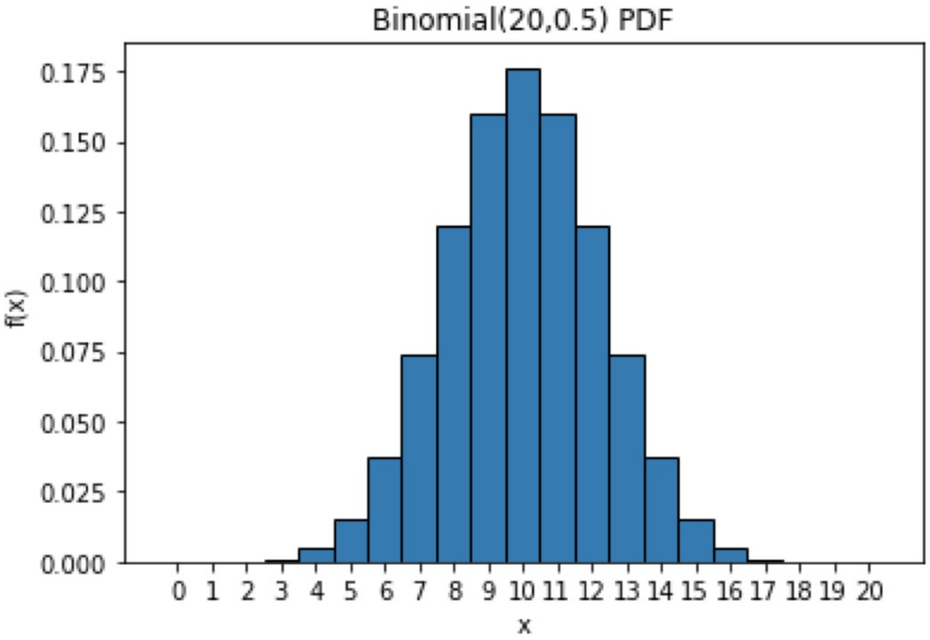
$$F_X(x) = \begin{cases} 0 & x < 0 \\ \frac{x}{10} & 0 \leq x \leq 10 \\ 1 & x > 10 \end{cases}$$

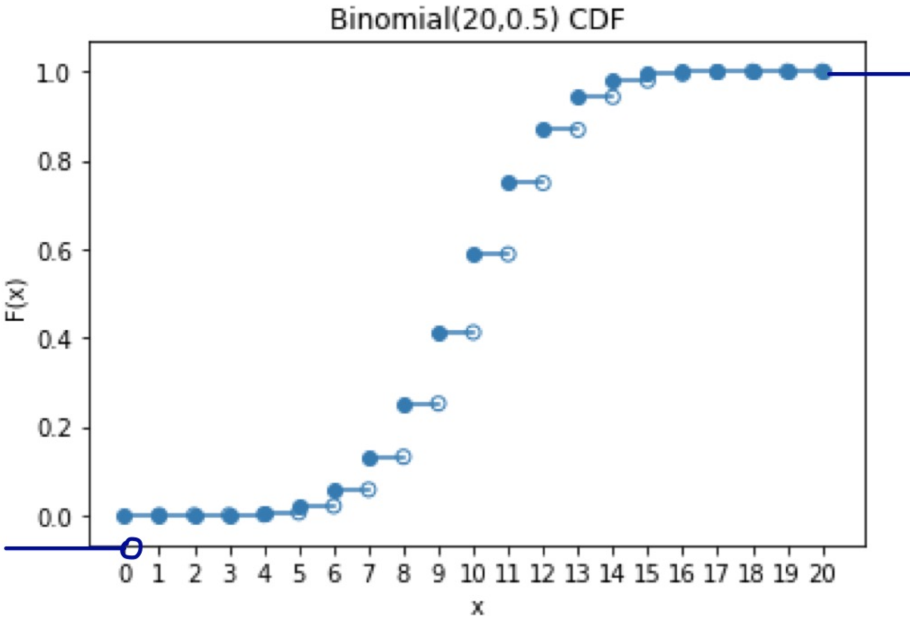
$$\Pr(6 \leq X \leq 8) = F_X(8) - F_X(6) = \frac{8}{10} - \frac{6}{10} = \frac{1}{5}$$



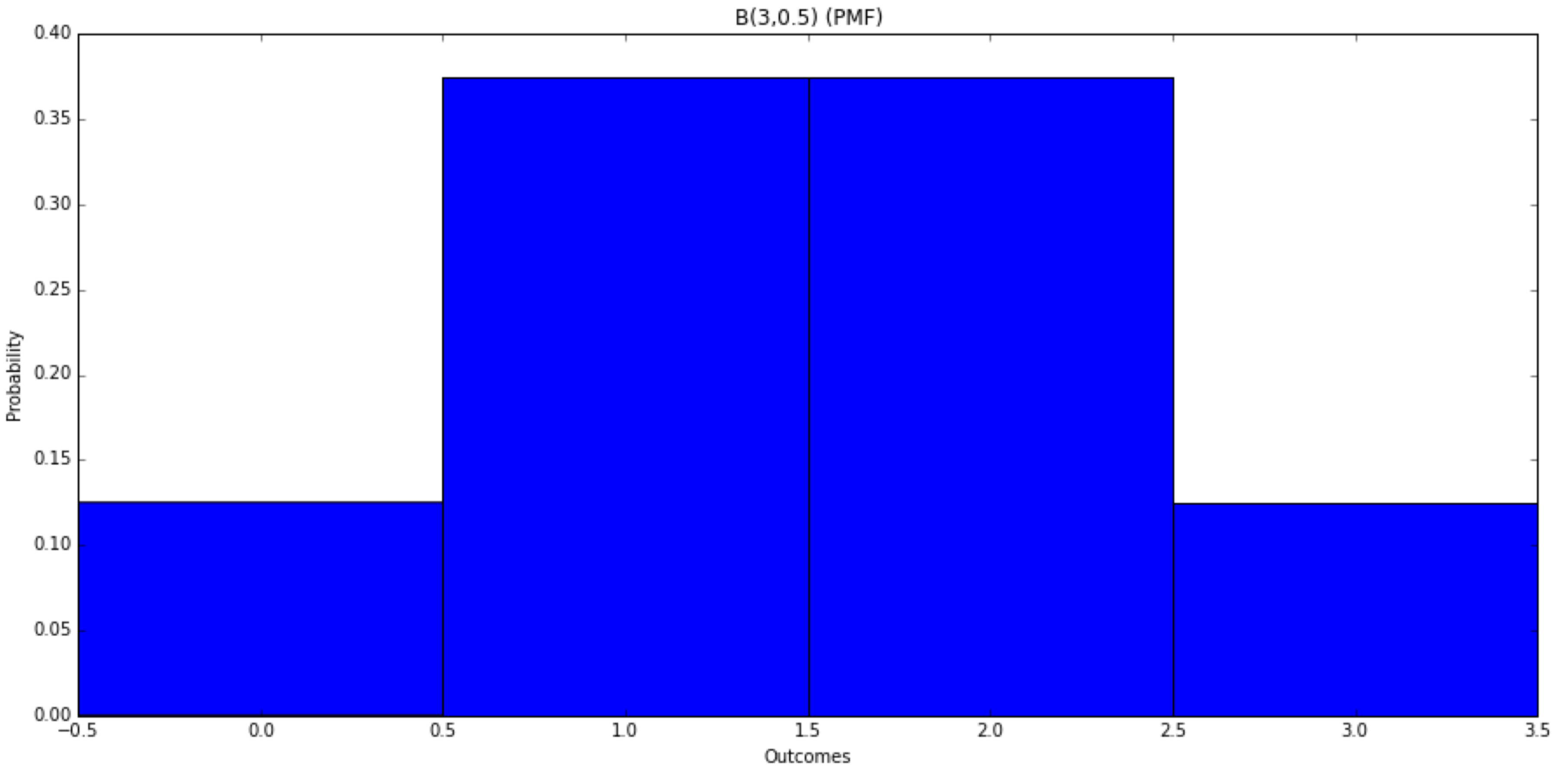
Recall: Binomial Distribution

- X ~ Binomial(n, p) example

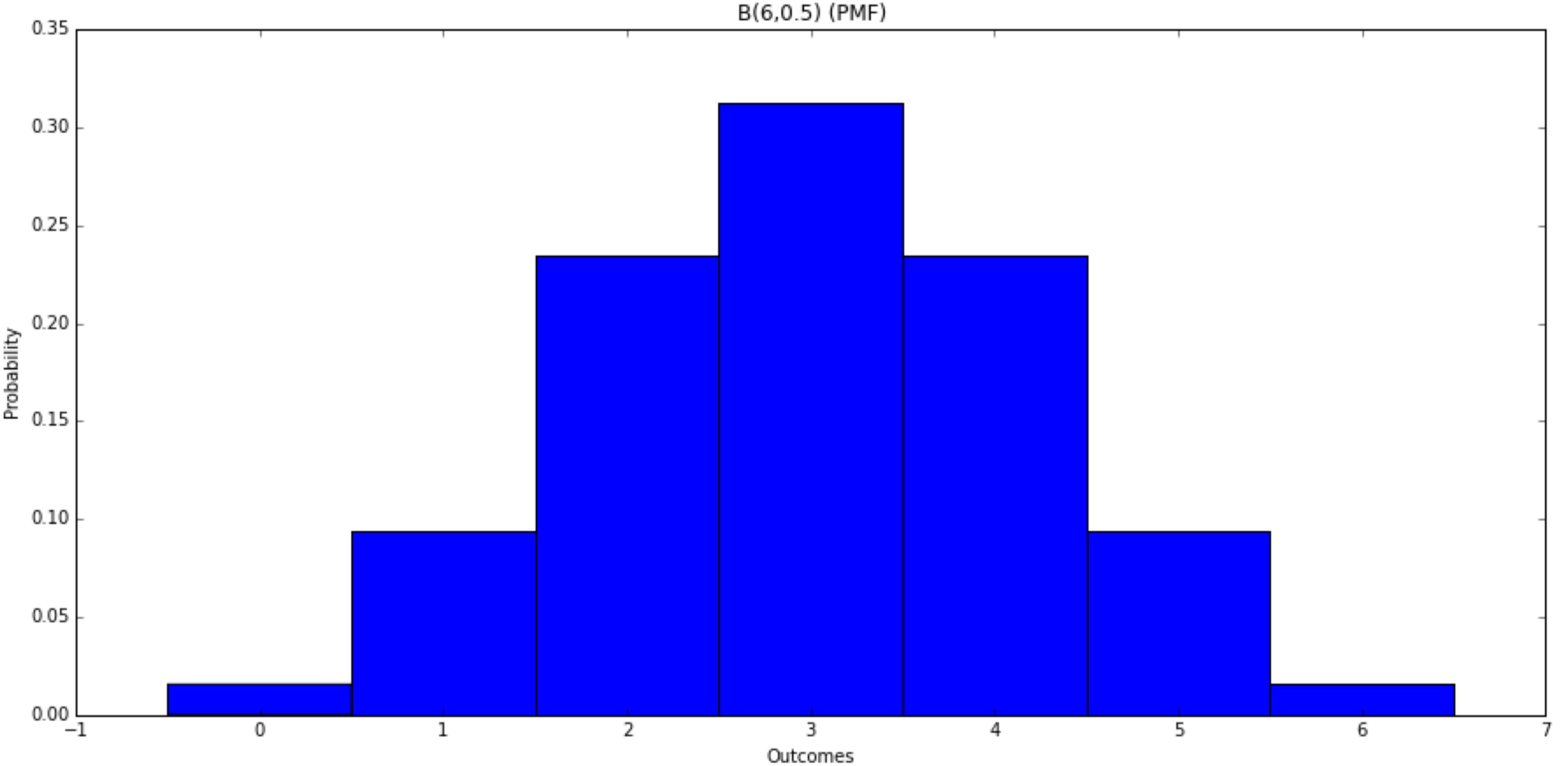




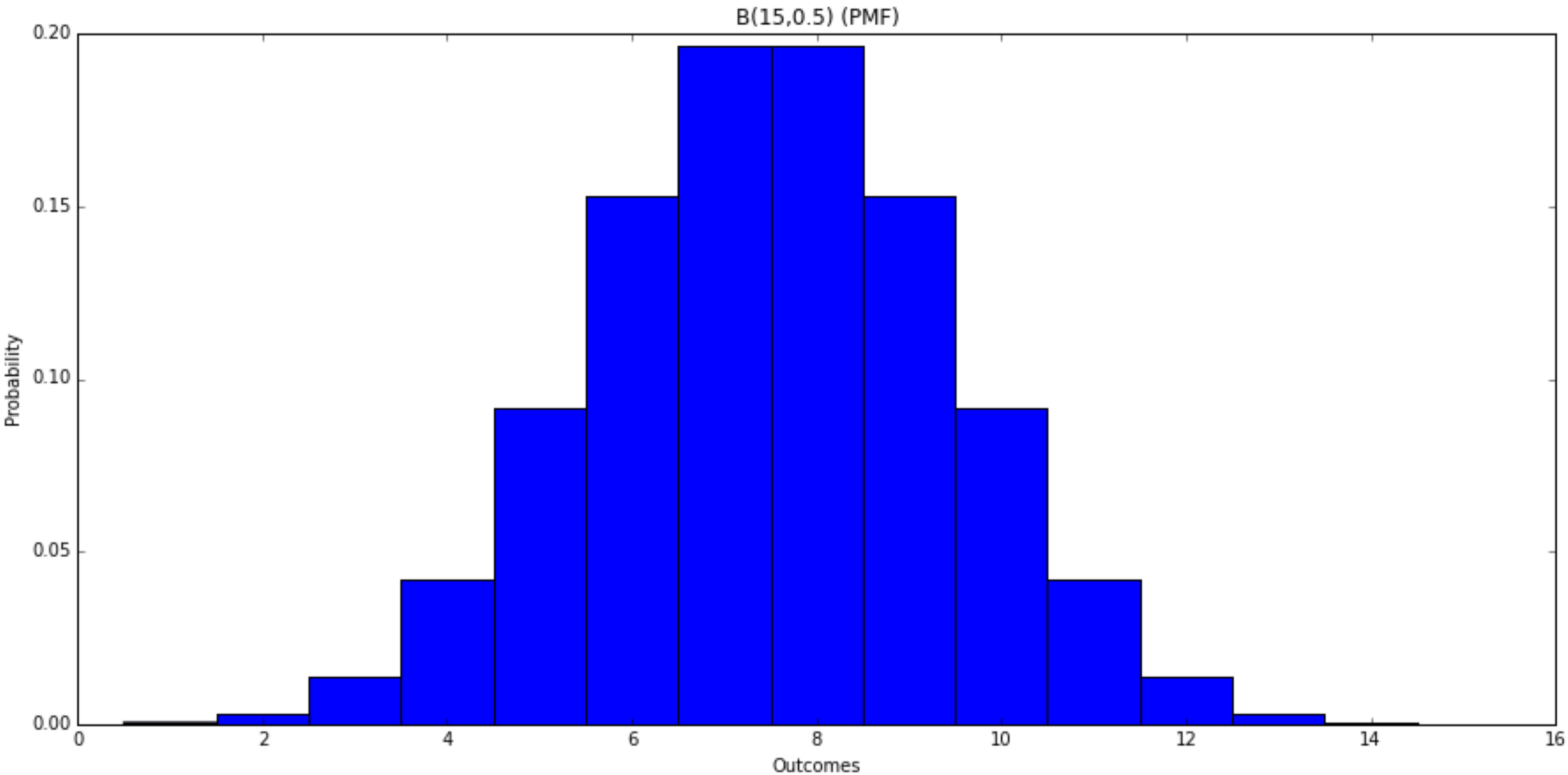
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



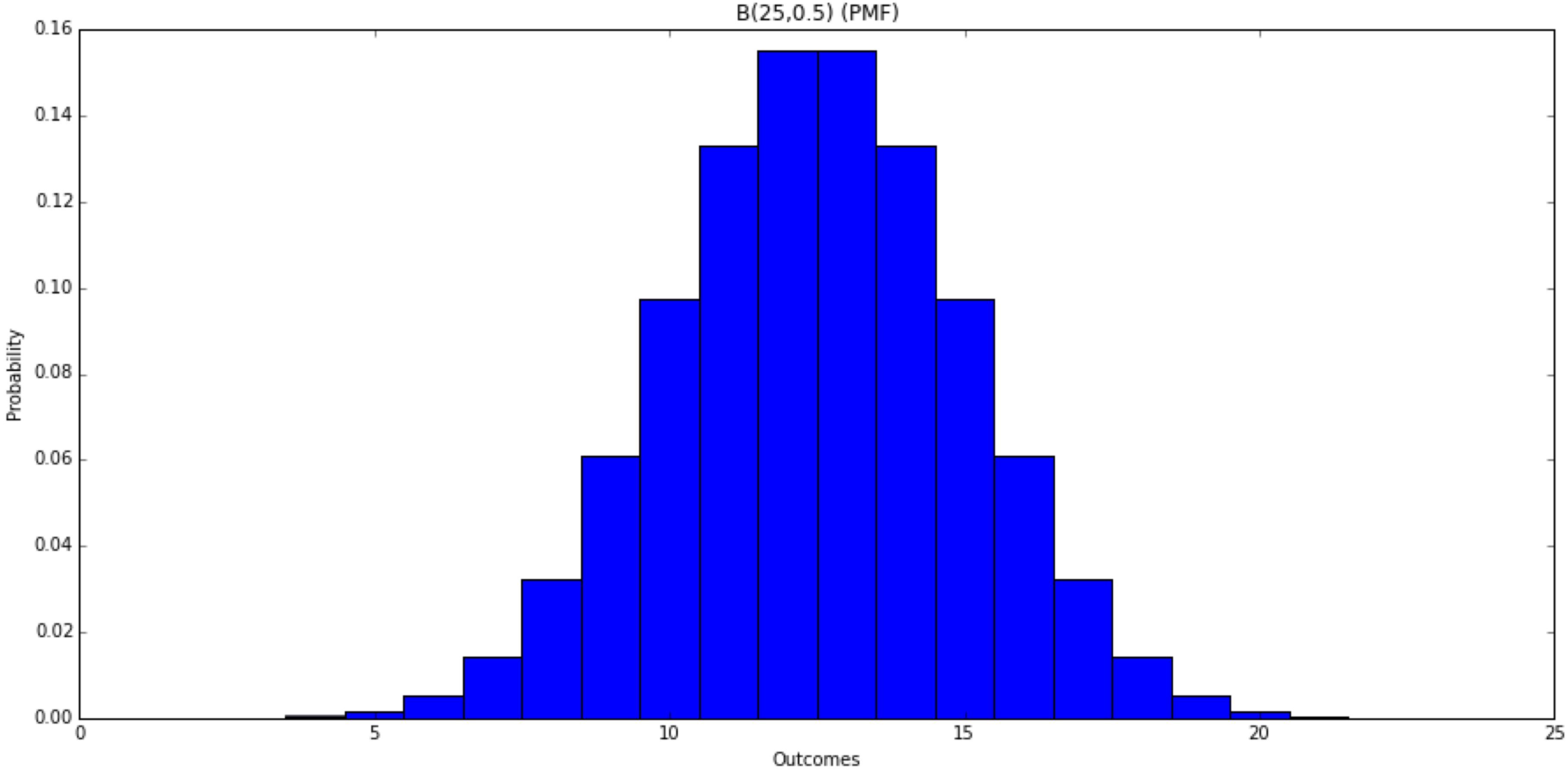
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



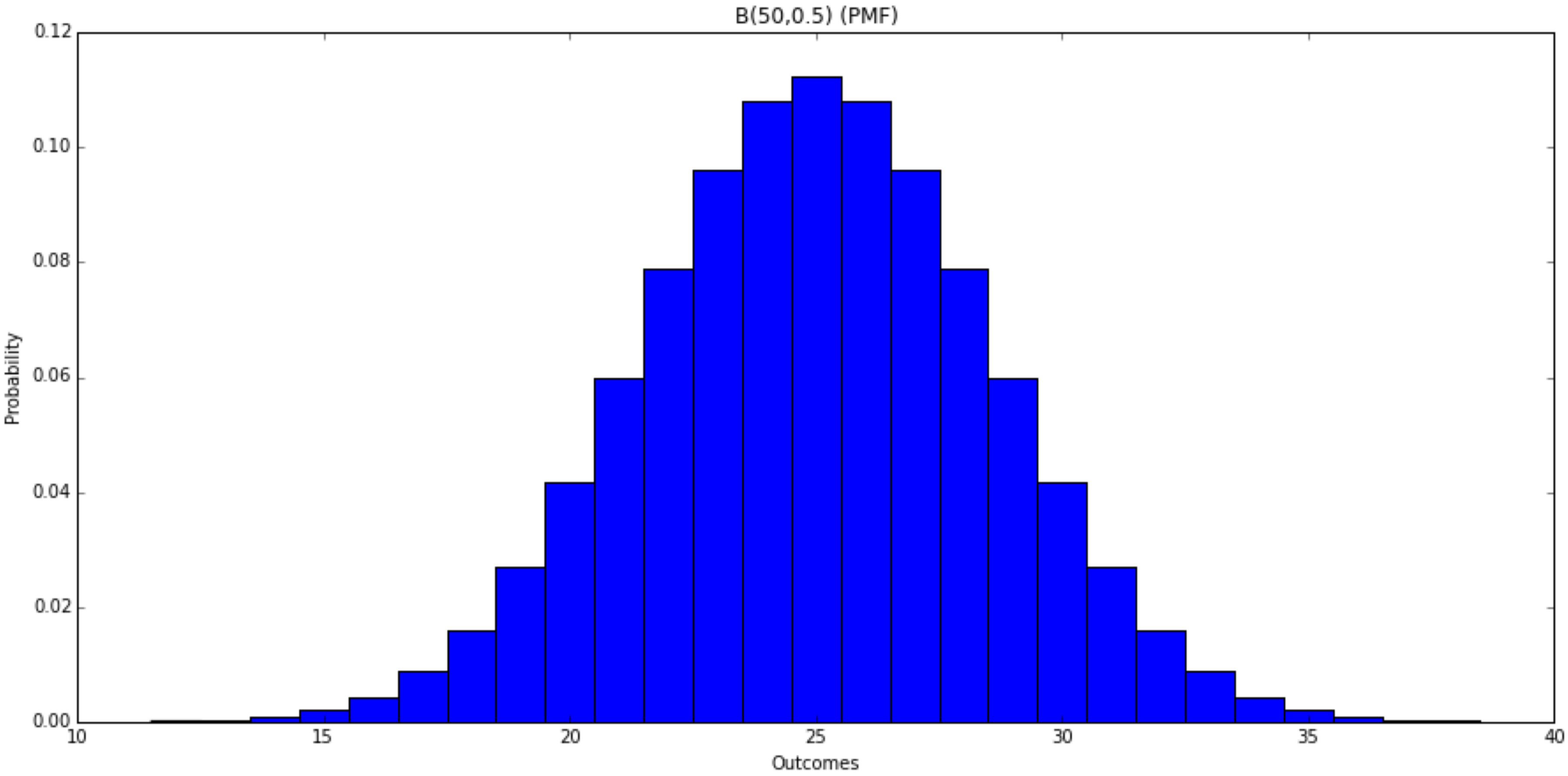
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



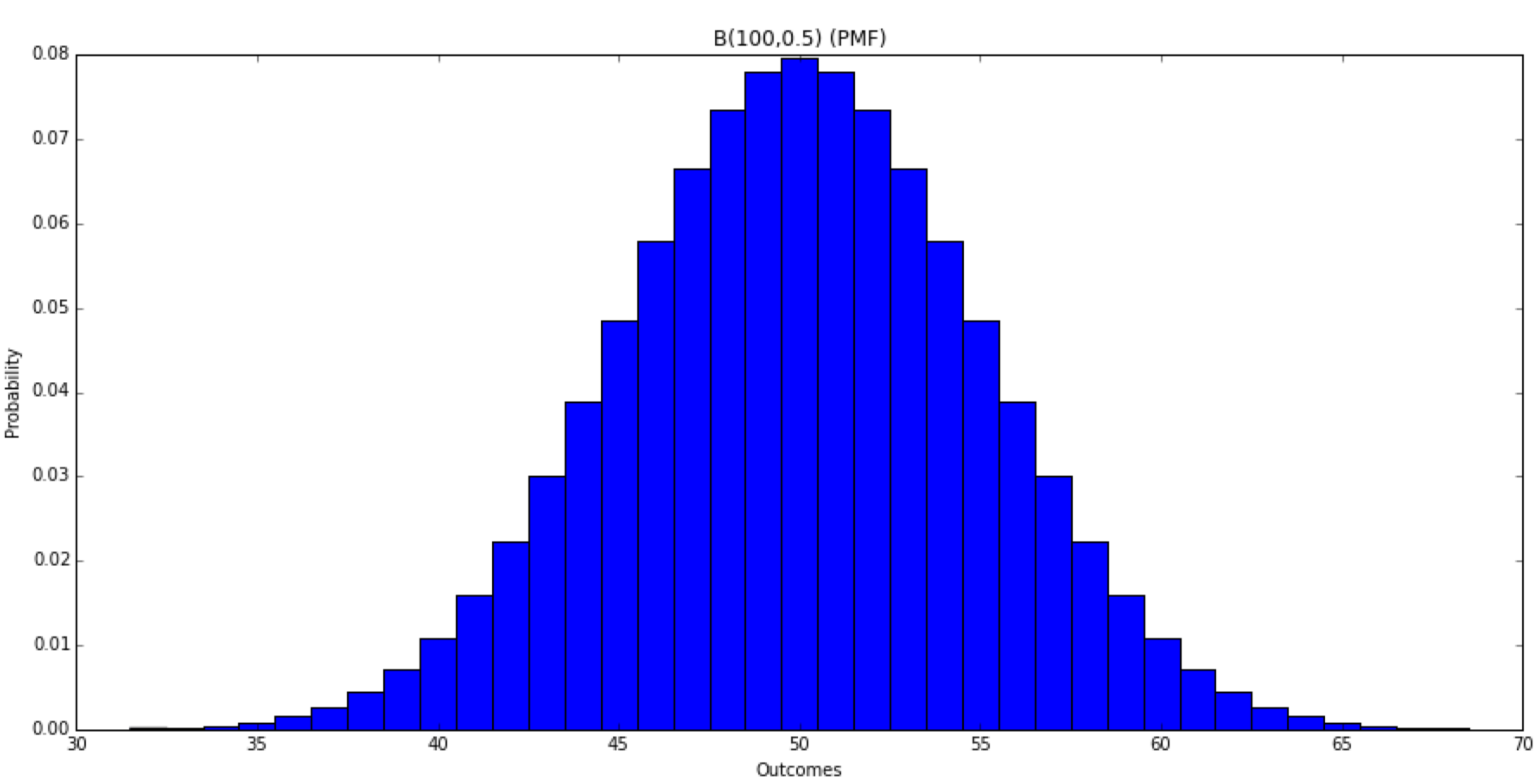
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



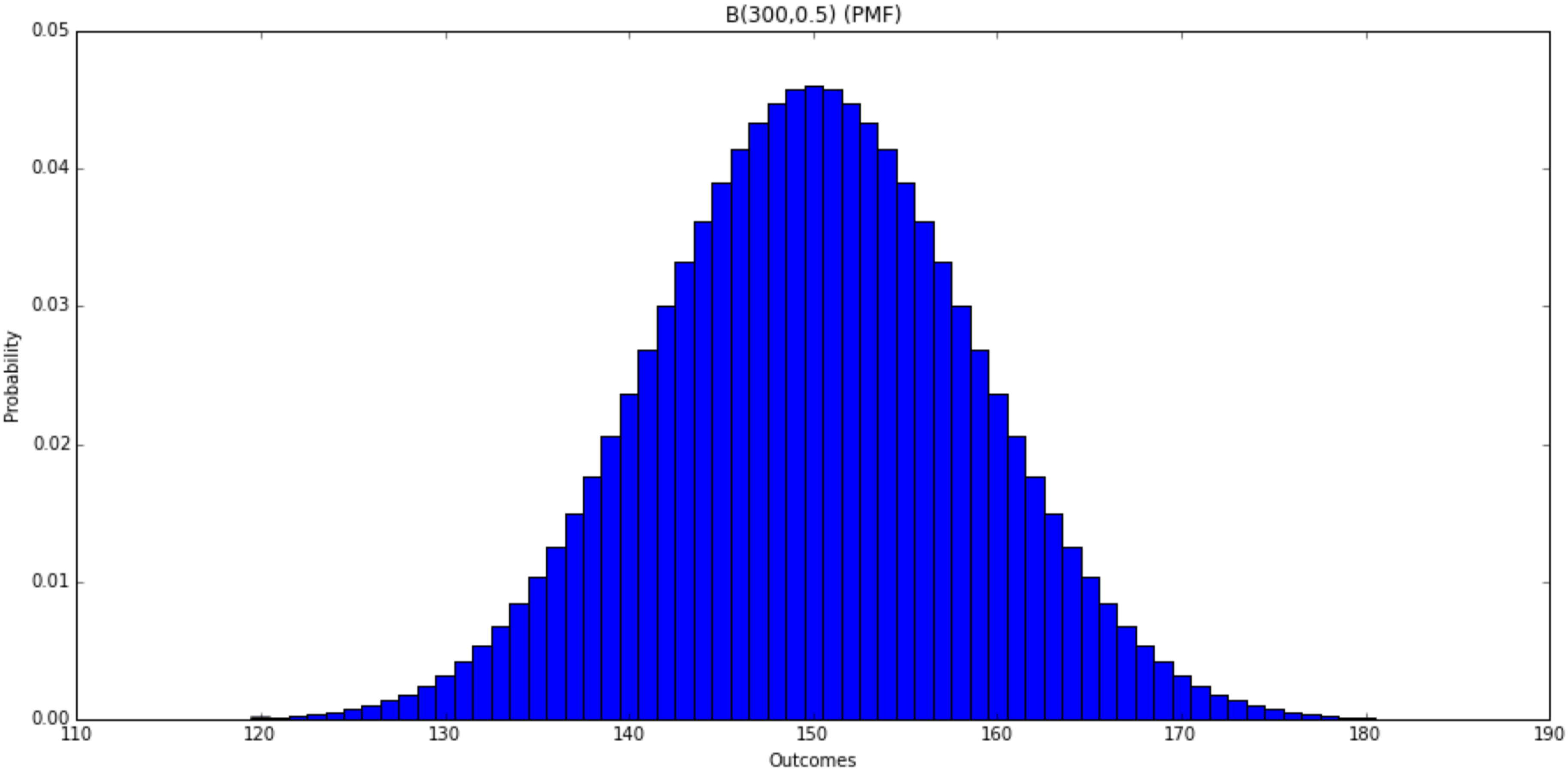
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



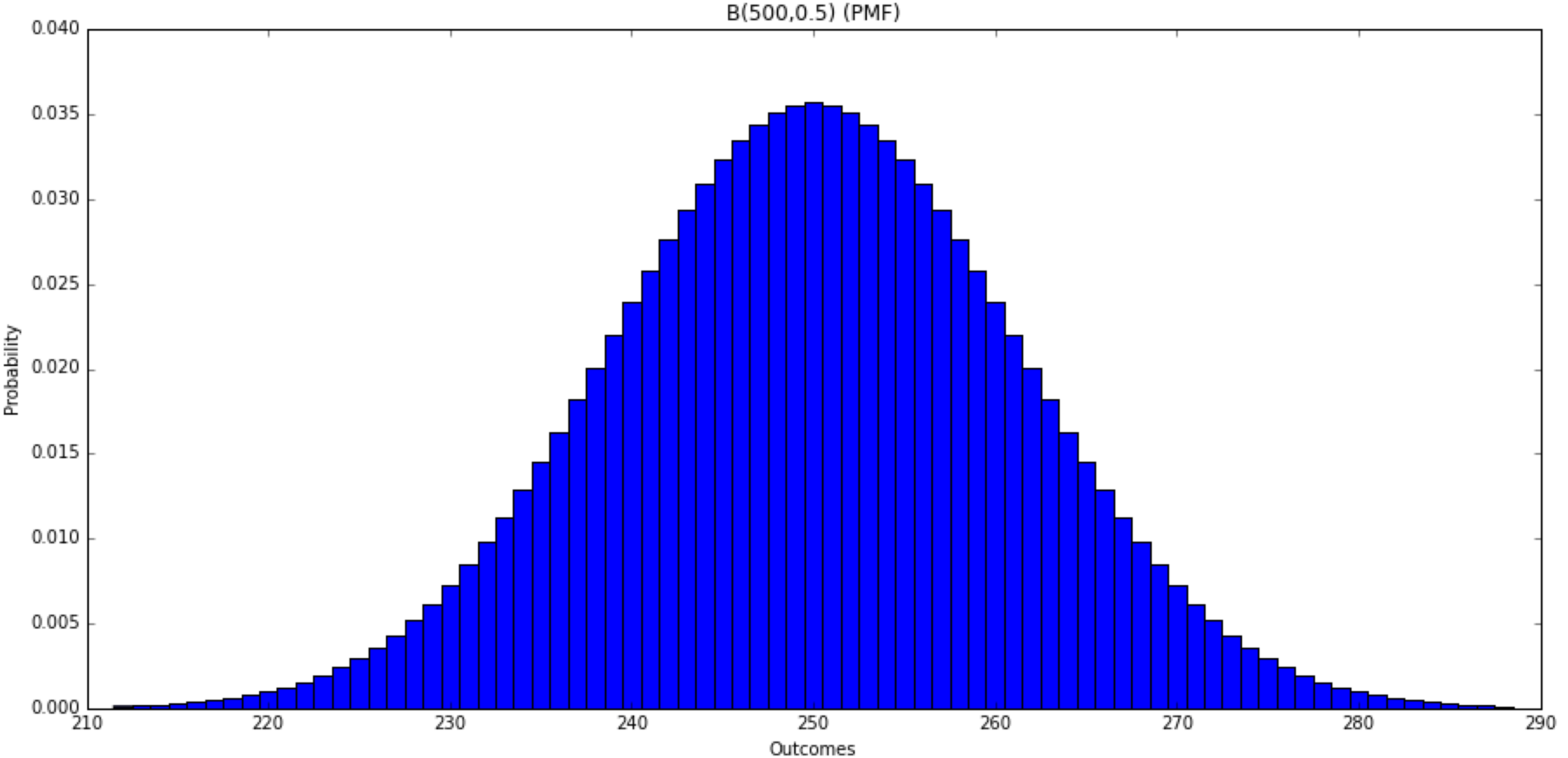
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



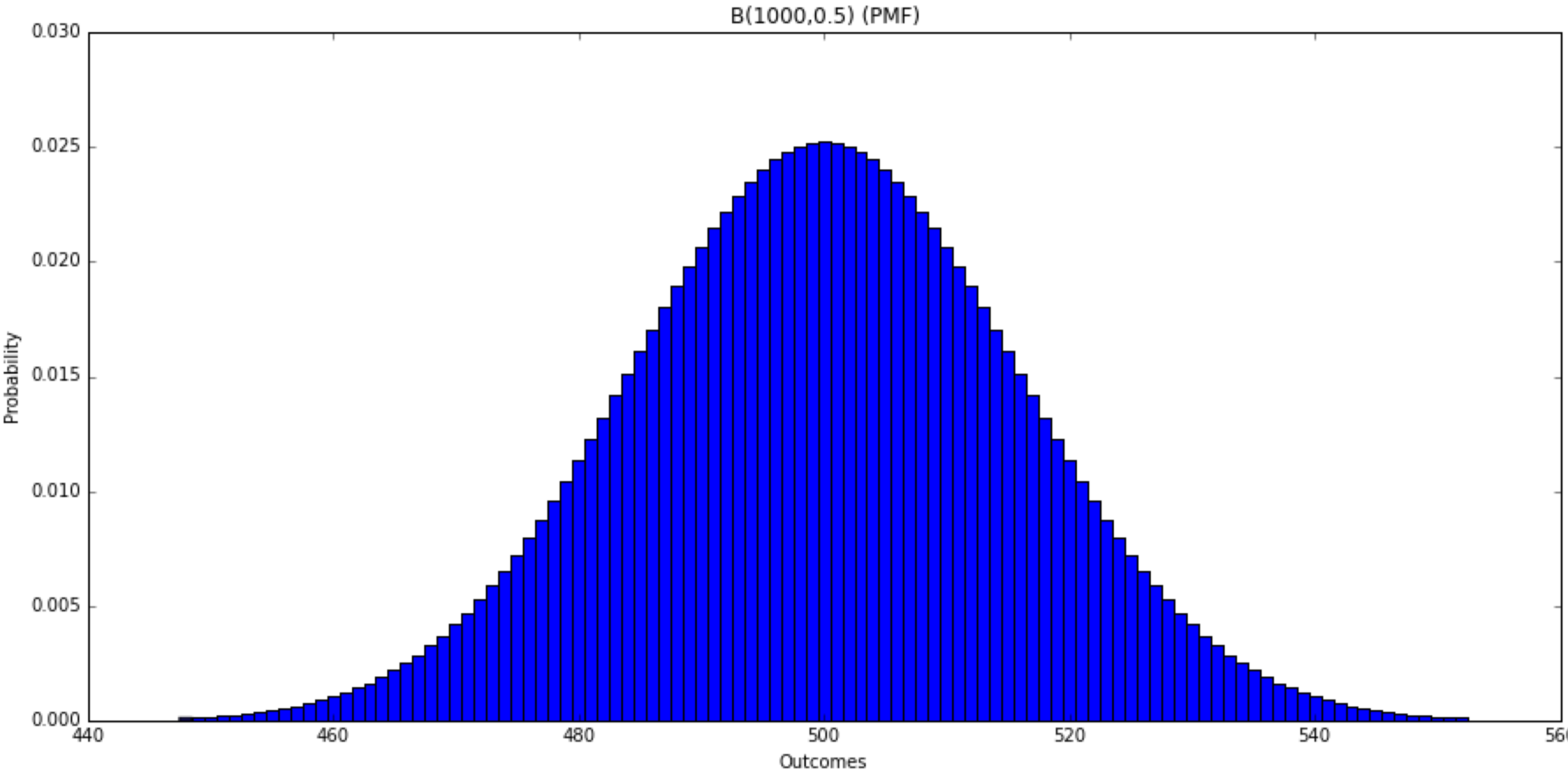
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



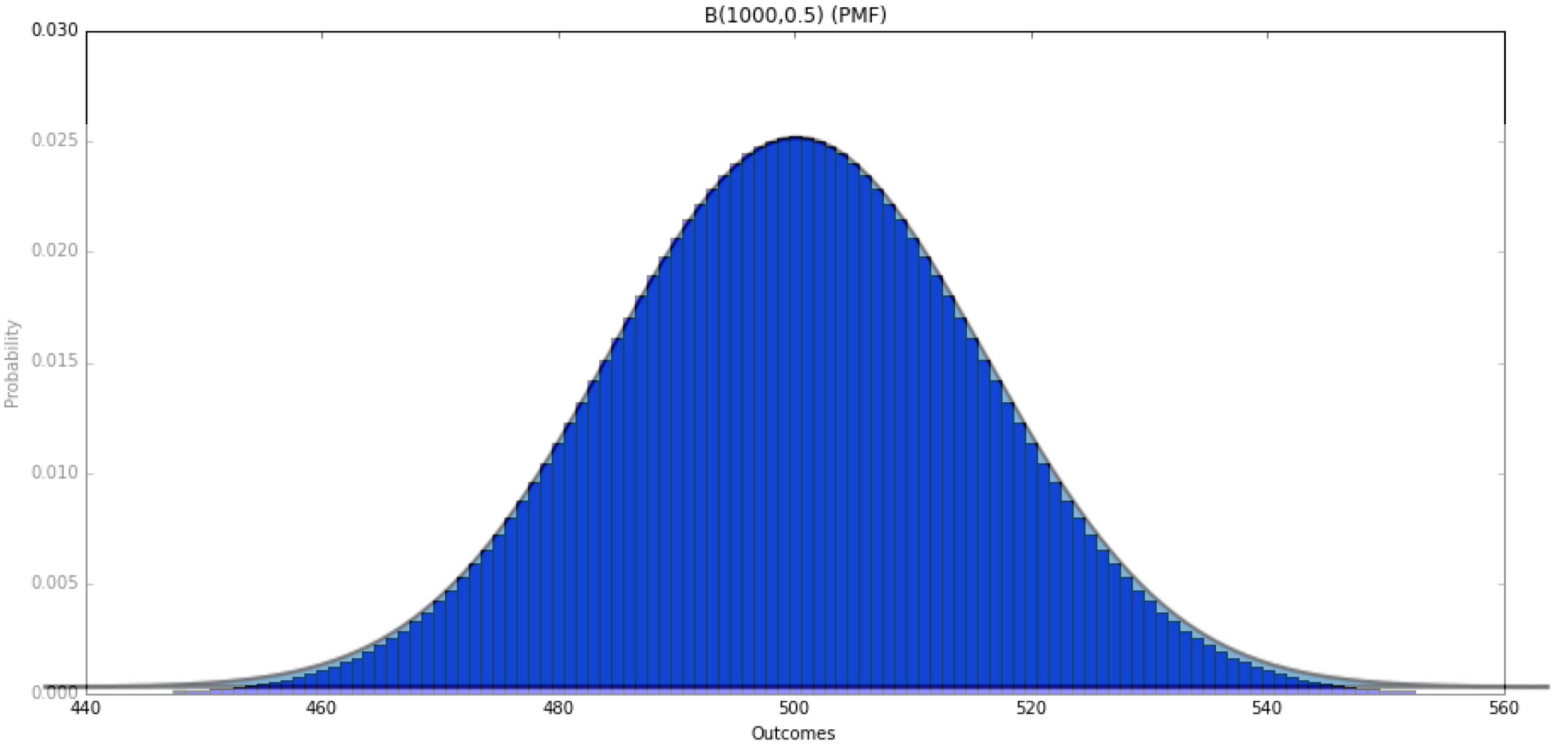
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



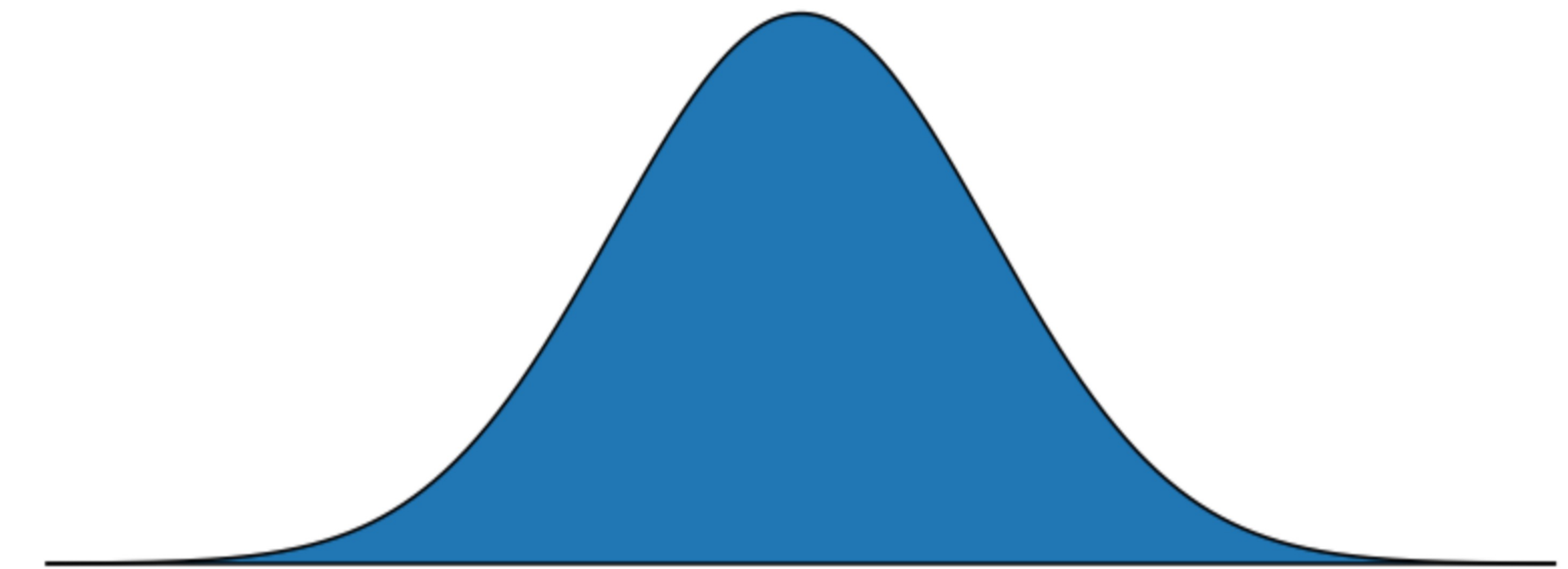
$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$



$X \sim Binomial(n, p)$ for $p = 0.5$ as $n \to \infty$

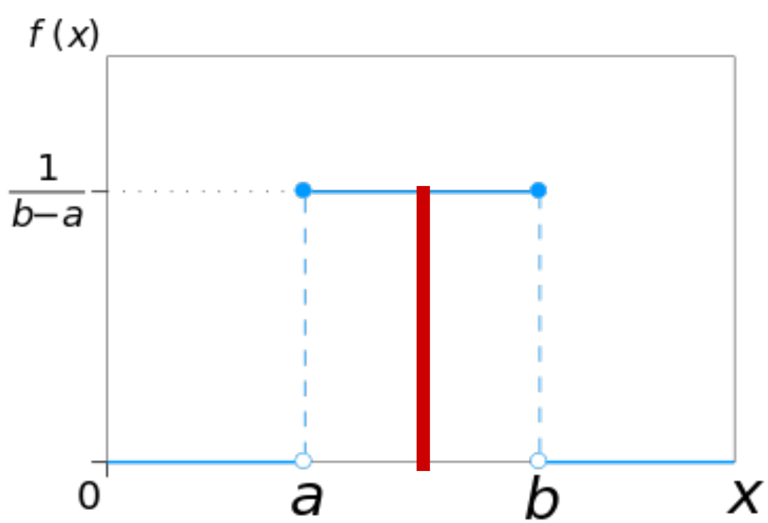


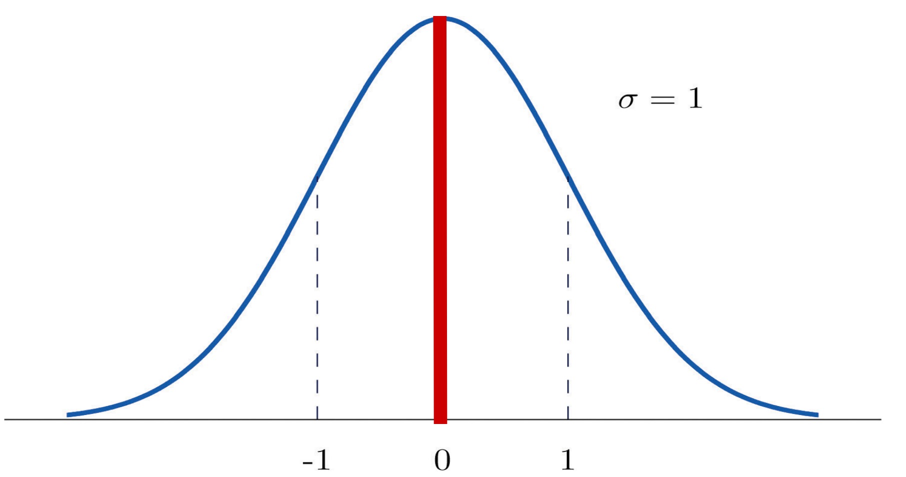
$X{\sim}Binomial(n,p)$ for $p = 0.5$ as $n \to \infty$



Symmetric probability distributions

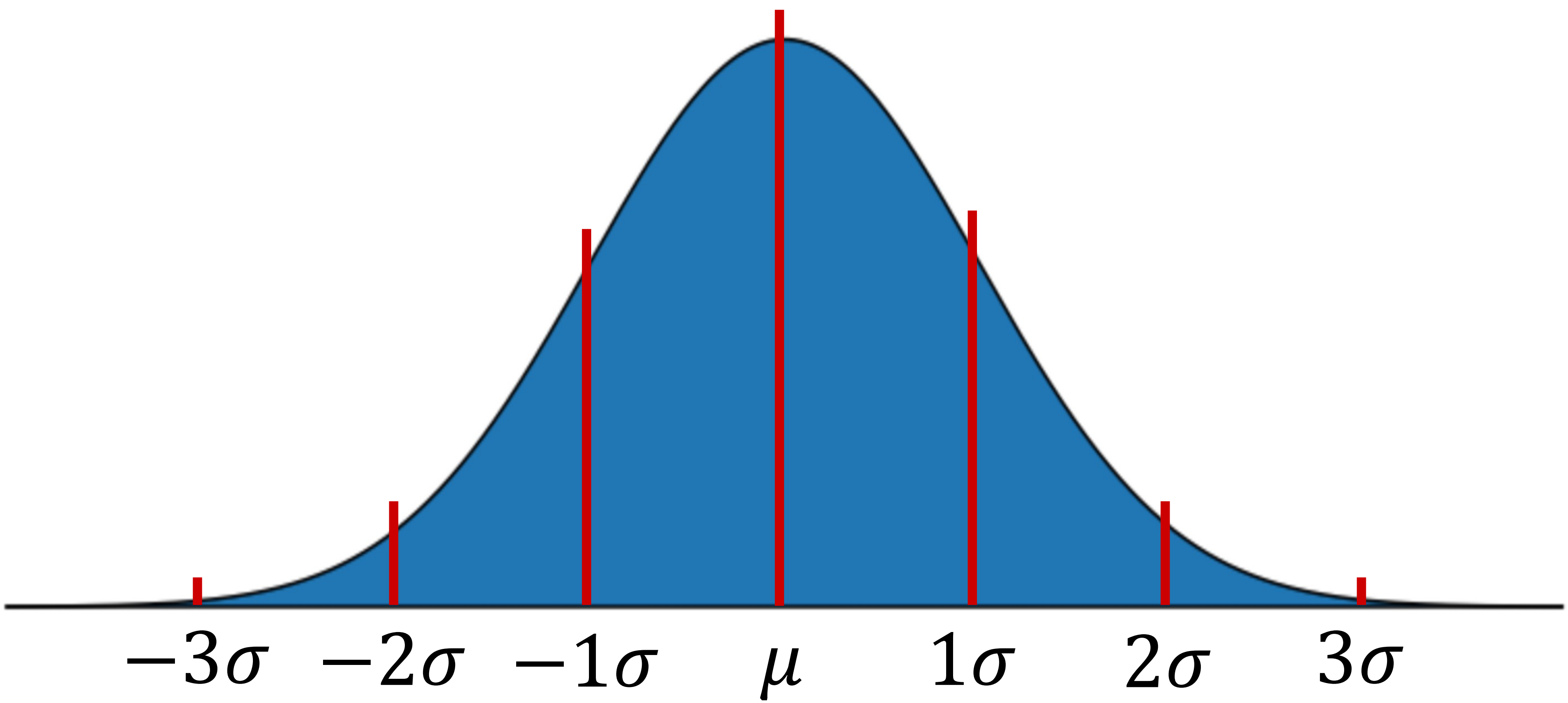
- A symmetric probability distribution is a probability distribution which is unchanged when its PDF is reflected around a vertical line at some value of the random variable represented by the distribution.



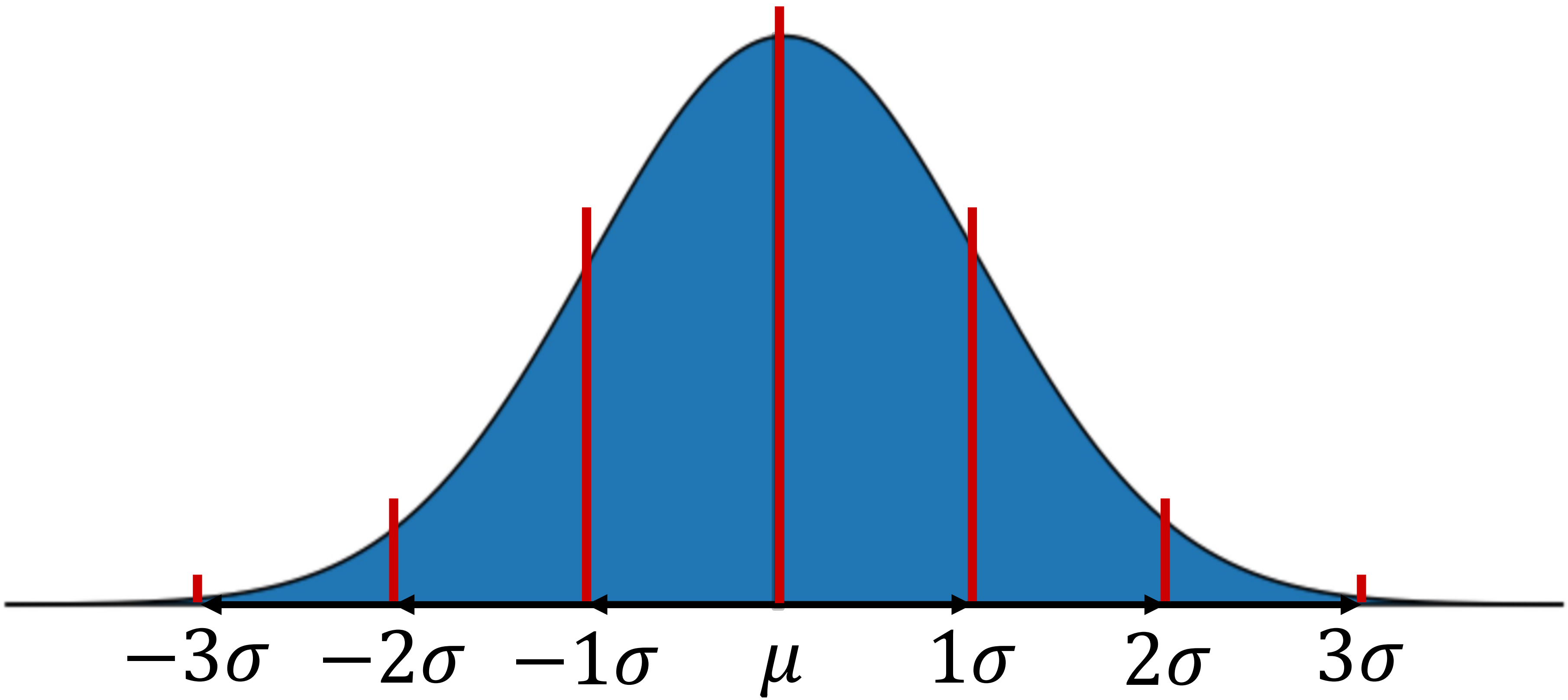


Normal Distribution

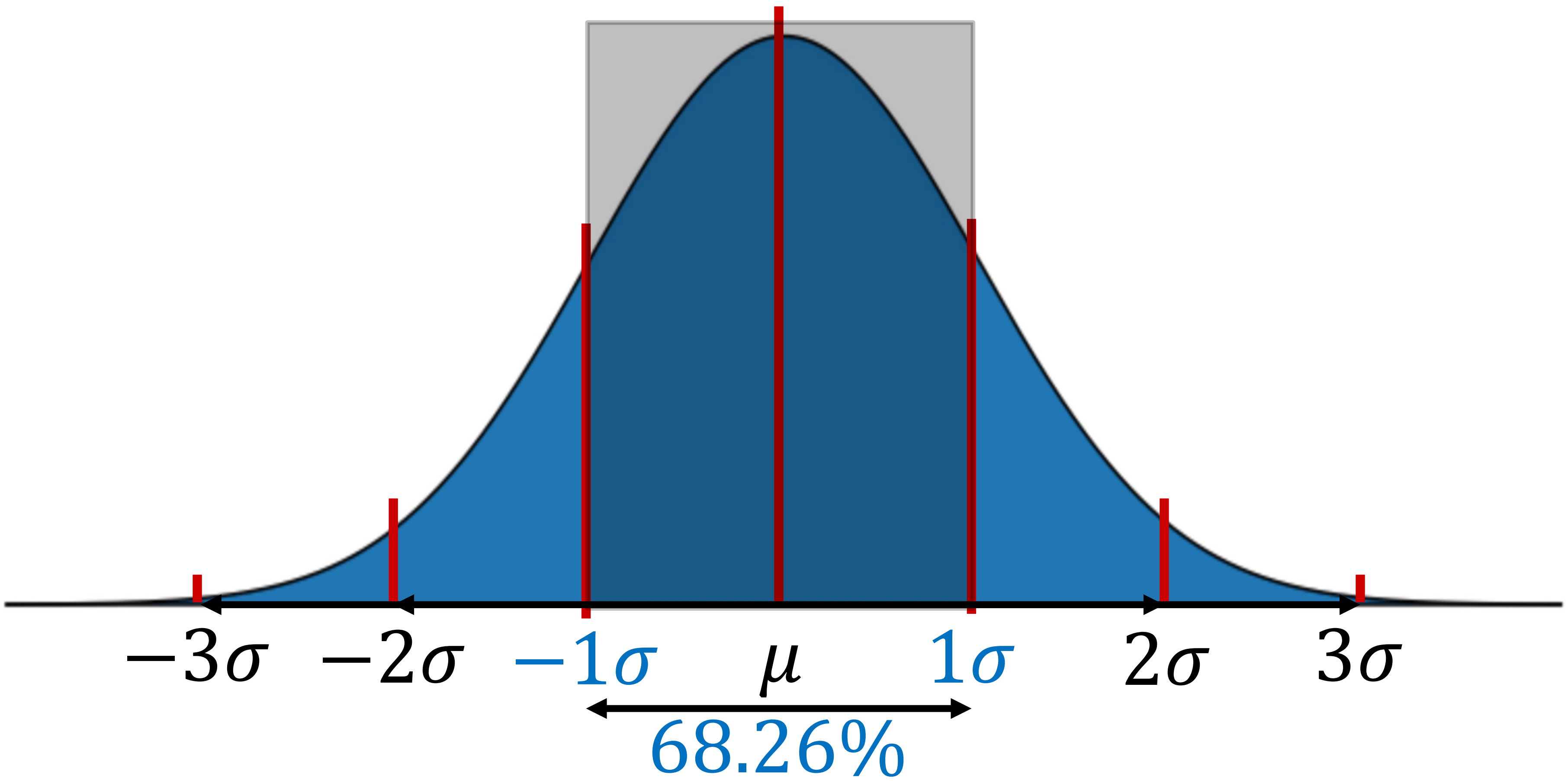
- Very important probability distribution
- Also known as Gaussian or Bell-shaped curve



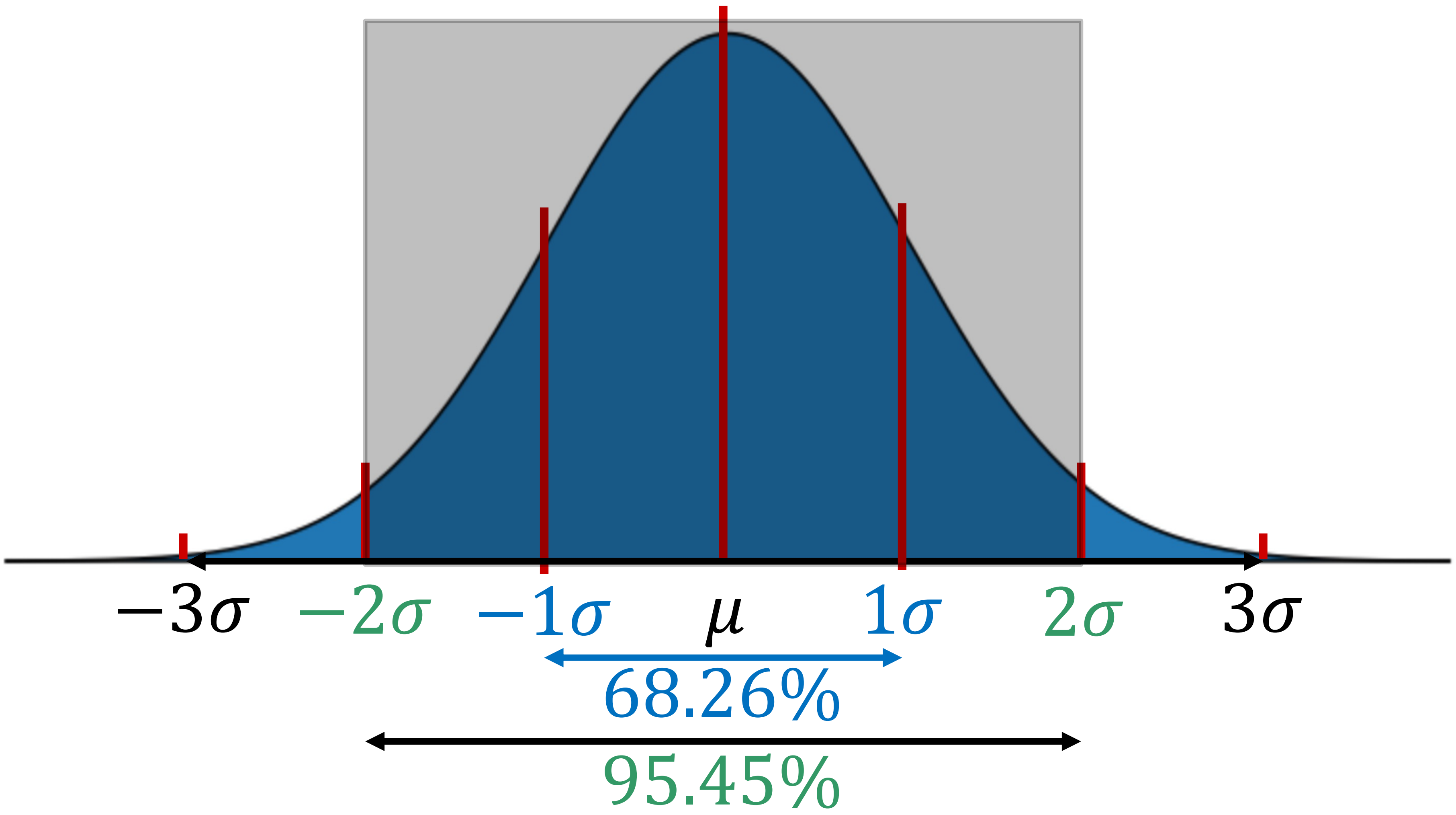
68-95-99.7% rule



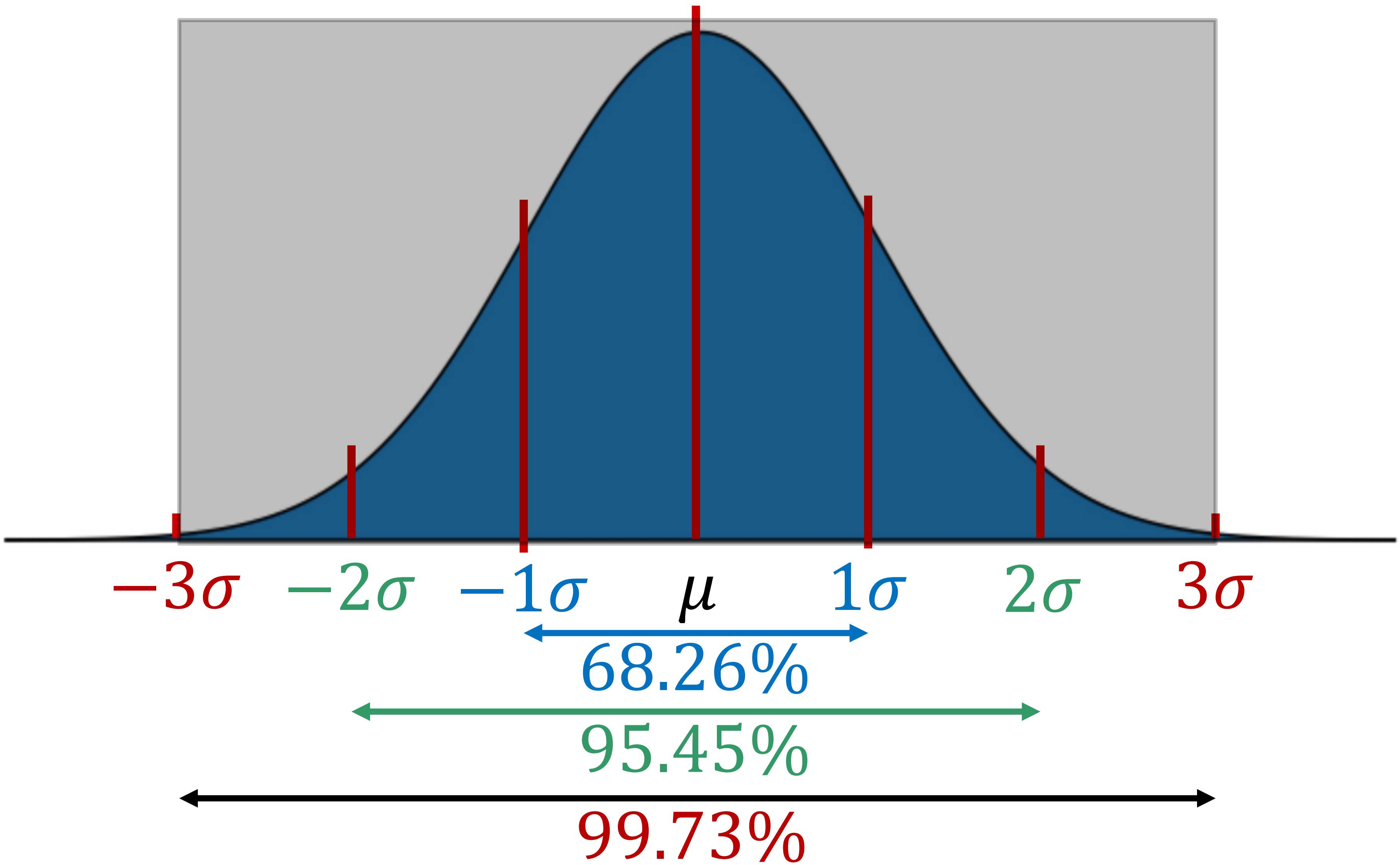
68-95-99.7% rule



68-95-99.7% rule



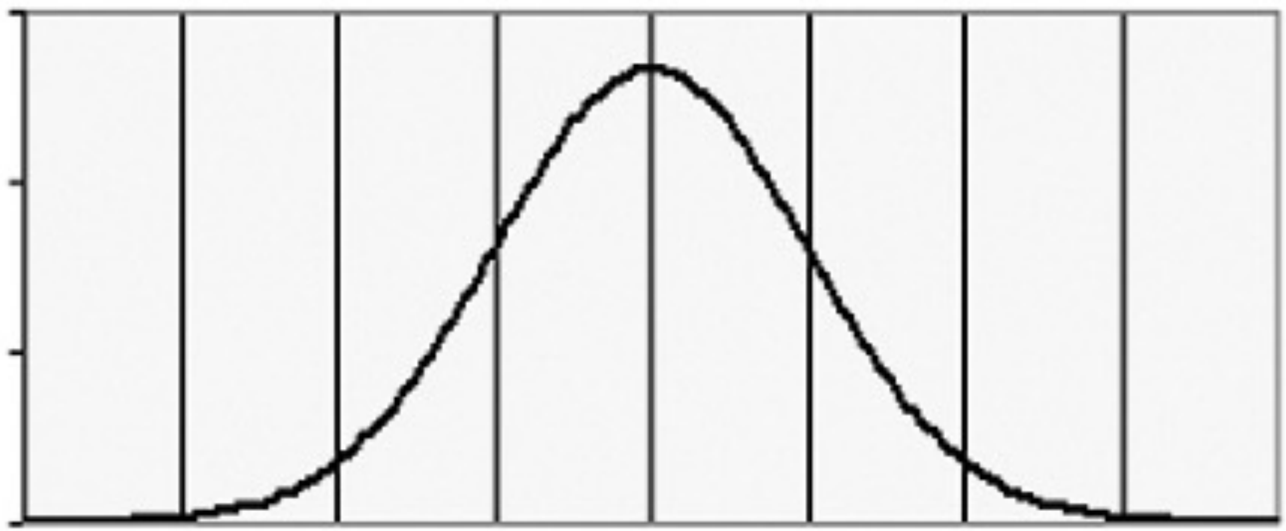
68-95-99.7% rule



Normal Distribution: Examples

The normal distribution occurs when a very large number of factors add together to create some random phenomenon.

Example: What is the IQ of a human being?

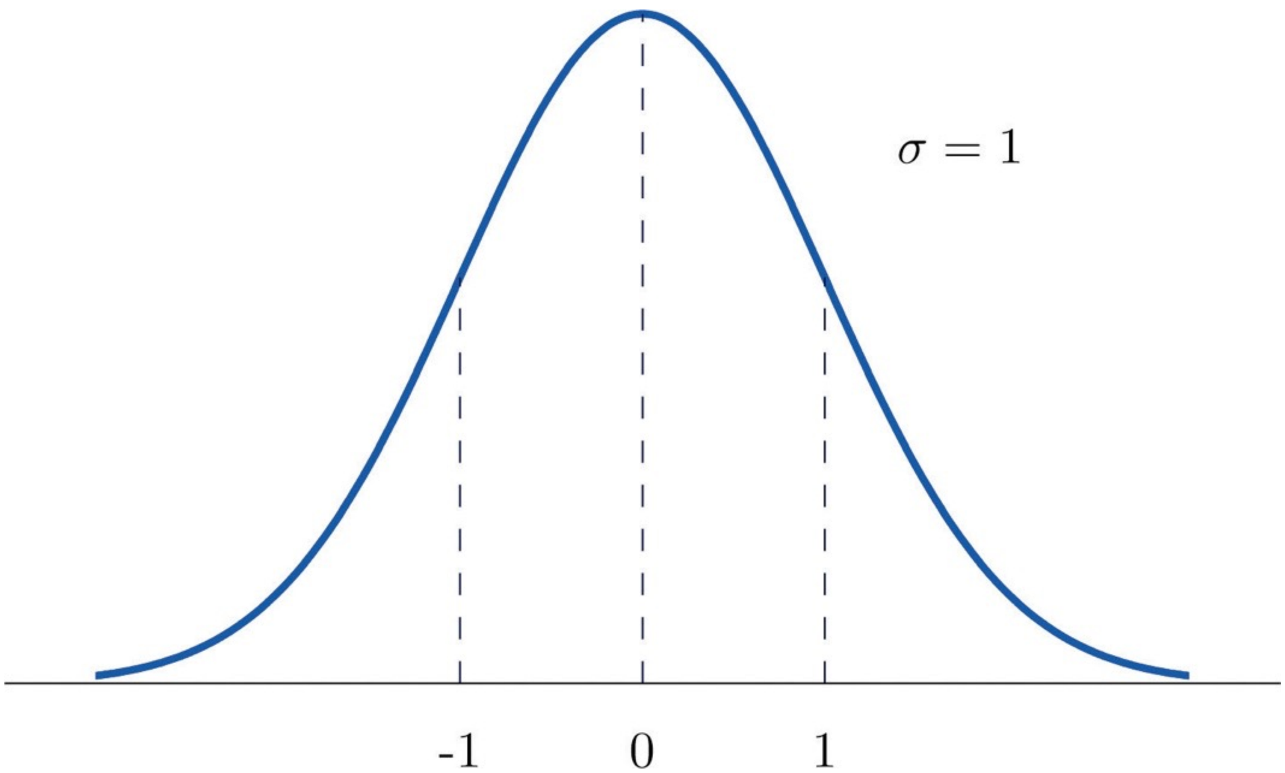


Standard Deviations	-4	-3	-2	-1	0	1	2	3	4
Wechsler IQ	40	55	70	85	100	115	130	145	160
Stanford-Binet IQ	36	52	68	84	100	116	132	148	164
Cumulative %	0.003	0.135	2.275	15.866	50.000	84.134	97.725	99.865	99.997

Standard Normal Distribution

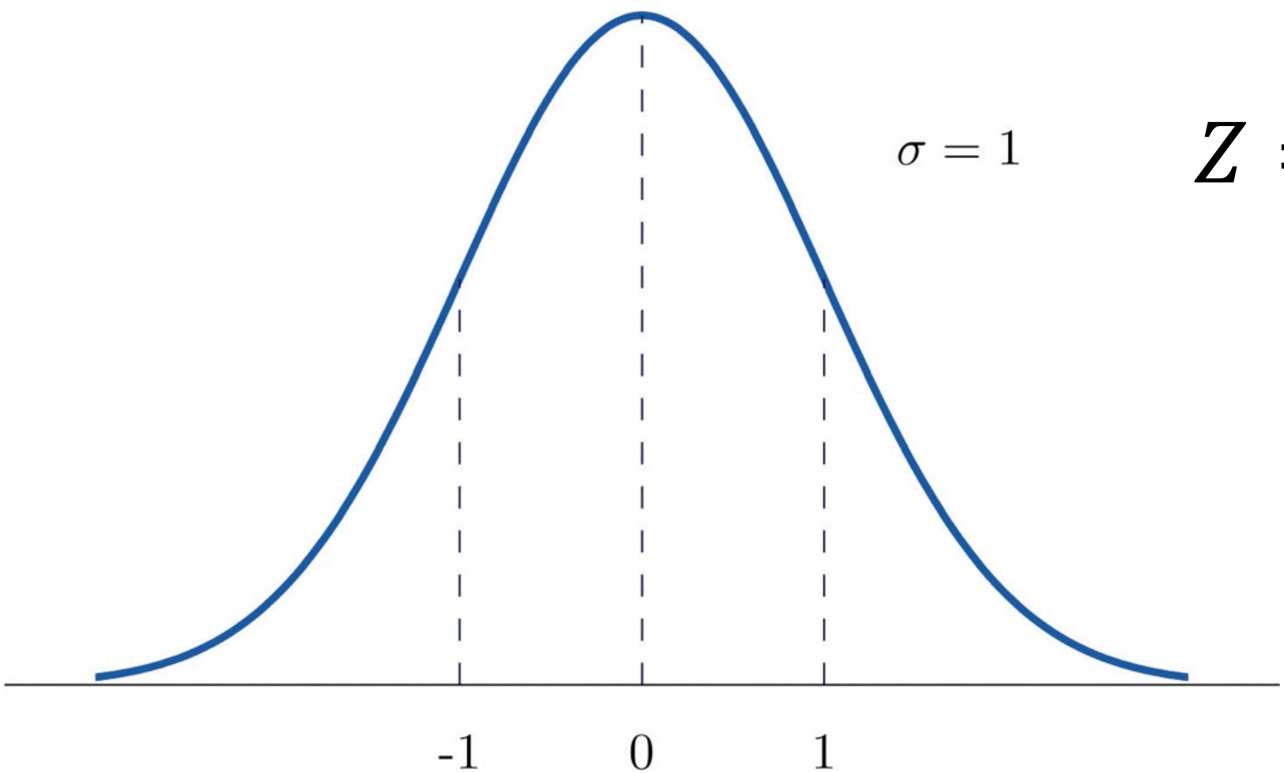
Since there are countless Normal Distributions, we focus on a normalized version, simply called the

Standard Normal Distribution



Standard Normal Distribution

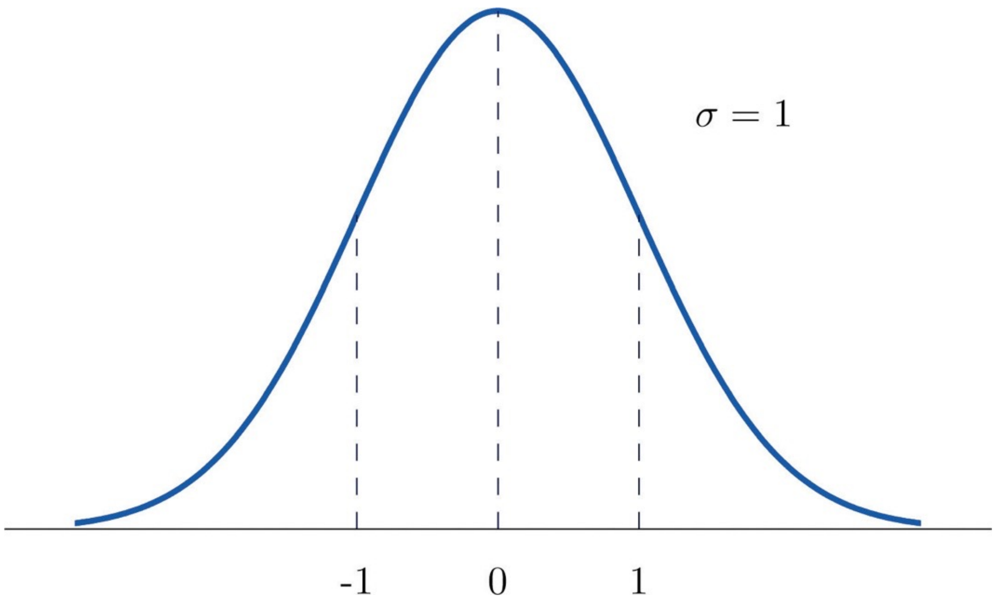
We can convert any random variable which has a normal distribution $X \sim N(\mu, \sigma^2)$ into a standardized random variable $Z \sim N(0,1)$ by computing its z-score



$$Z = \frac{X - \mu_x}{\sigma_X}$$

Standard Normal Distribution

We can convert any random variable which has a normal distribution $X \sim N(\mu, \sigma^2)$ into a standardized random variable $Z \sim N(0,1)$ by computing its z-score

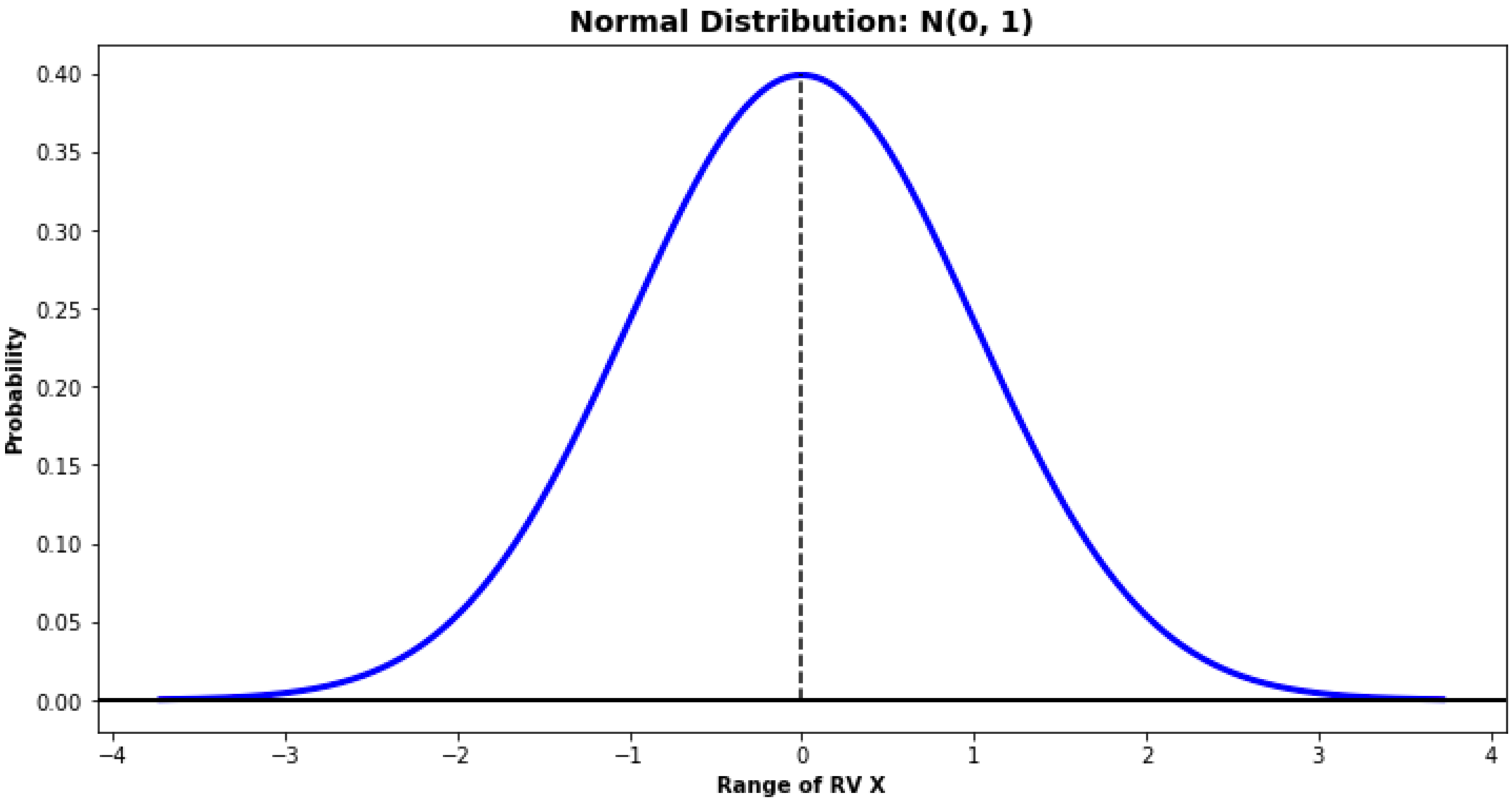


$$X = \sigma_X \cdot Z + \mu$$

Top Hat Activity (Join Code 033357)

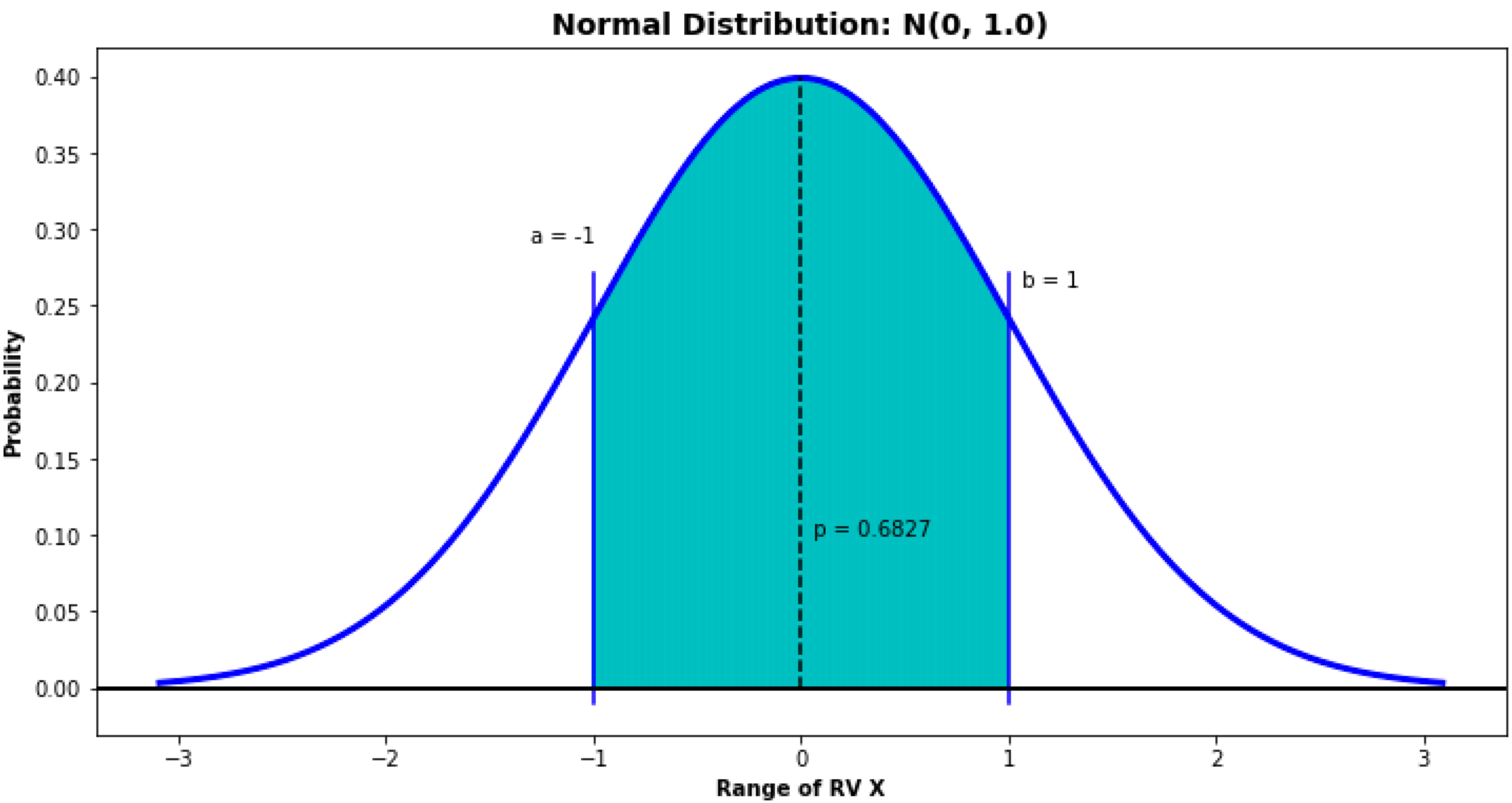
- The heights of adult men in the United States are approximately normally distributed with a mean of 70 inches and a standard deviation of 3 inches.
- Heights of adult women are approximately normally distributed with a mean of 64.5 inches and a standard deviation of 2.5 inches.
- Compute the z-score of your height

Normal Distribution



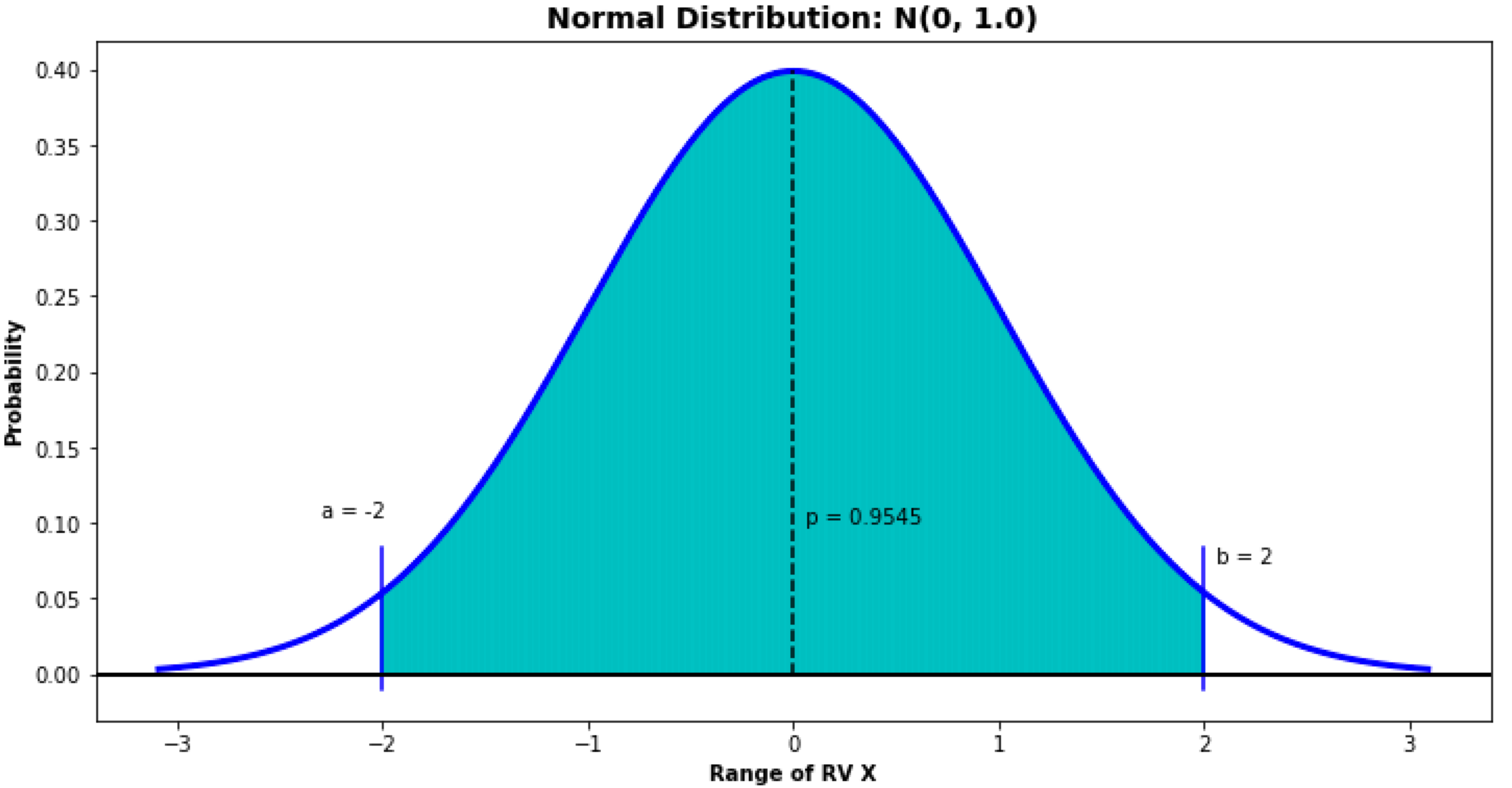

```
mean = 0
var = 1
stdev = 1
```

Normal Distribution



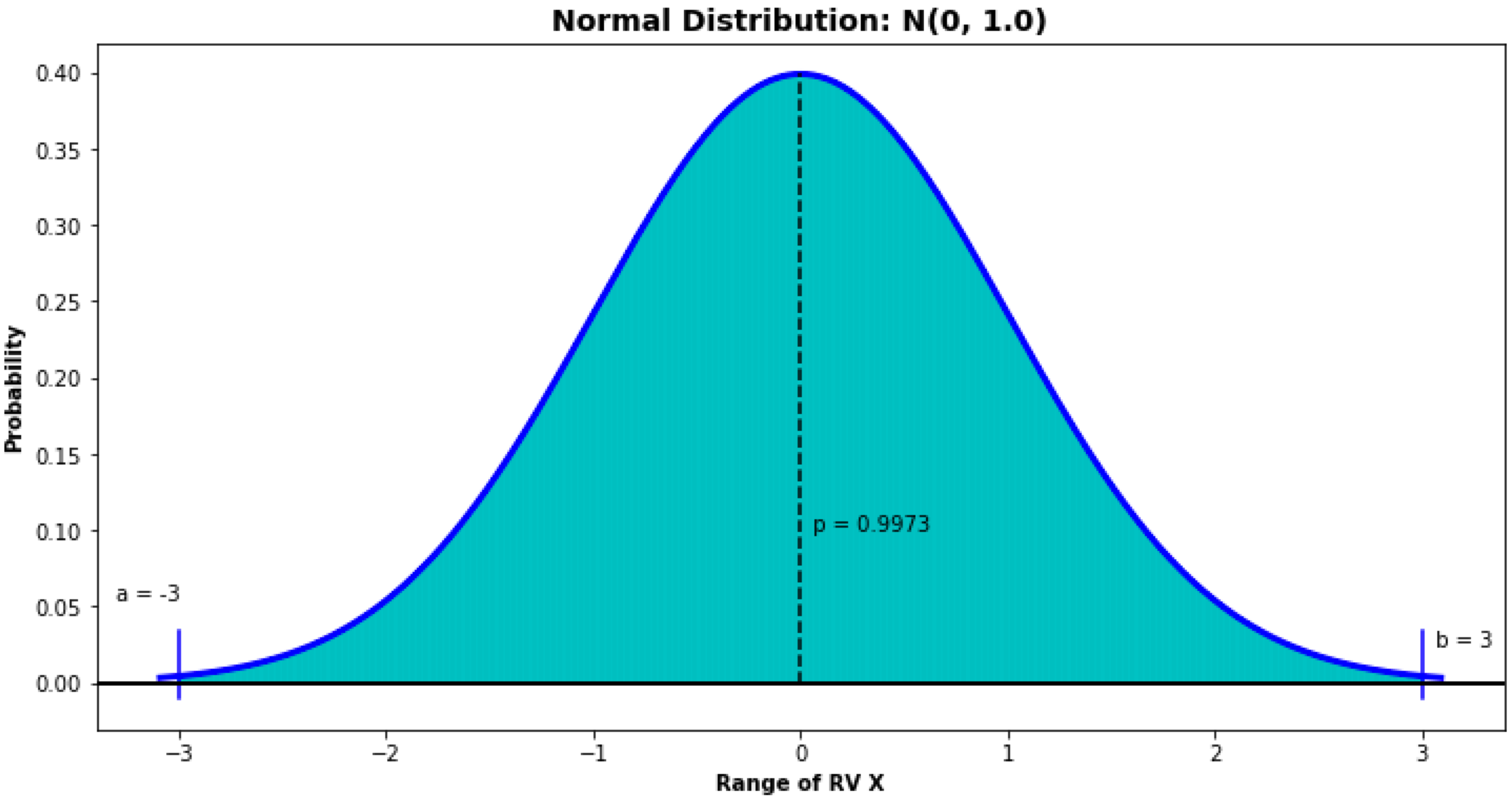

```
mean = 0
var = 1
stdev = 1.0
P(−1 < X < 1) = P(X < 1) − P(X < −1) = 0.8413 − 0.1587 = 0.6827
```

Normal Distribution




```
mean = 0
var = 1
stdev = 1.0
P(−2 < X < 2) = P(X < 2) − P(X < −2) = 0.9772 − 0.0228 = 0.9545
```

Normal Distribution




```
mean = 0
var = 1
stdev = 1.0
P(−3 < X < 3) = P(X < 3) − P(X < −3) = 0.9987 − 0.0013 = 0.9973
```

Example: Normal Distribution

Your local pizza shop claims that their large is at least 16in or it is free. Their pizza is normally distributed with $\mu = 16.3in$ and $\sigma = 0.2in$. What is the probability of getting free pizza?

PDF: Normal Distribution

- The PDF of a *standard normal* random variable, $Z \sim N(0,1)$ is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \text{for all } z \in \mathbb{R}.$$

- and its CDF is given by

$$F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{z} e^{-\frac{u^2}{2}} du$$

Unfortunately, this integral does not have a closed form solution ☹

PDF: Normal Distribution

- The PDF of a *standard normal* random variable, $Z \sim N(0,1)$ is given by

$$f_Z(z) = \frac{1}{\sqrt{2\pi}} e^{-\frac{z^2}{2}}, \text{for all } z \in \mathbb{R}.$$

- and its CDF is given by

$$\Phi(x) = \Pr(Z \leq x) = F_Z(z) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-\frac{u^2}{2}} du$$

Unfortunately, this integral does not have a closed form solution ☹

Properties of $\Phi(x)$

Here are some cool properties of ϕ

1. $\lim_{x\to\infty} \Phi(x) = 1$ and $\lim_{x\to-\infty} \Phi(x) = 0$
2. $\Phi(0) = \frac{1}{2}$
3. $\Phi(-x) = 1 - \Phi(x)$, for all $x \in \mathbb{R}$

If Z is a standard normal variable and $X = \sigma Z + \mu$ is a normal random variable, then

$$X \sim N(\mu, \sigma^2)$$

Getting Free Pizza

Your local pizza shop claims that their large is at least 16in or it is free. Their pizza is normally distributed with $\mu = 16.3in$ and $\sigma = 0.2in$. What is the probability of getting free pizza?

Getting Huge Pizza

Your local pizza shop claims that their large is at least 16in or it is free. Their pizza is normally distributed with $\mu = 16.3in$ and $\sigma = 0.2in$. What is the probability of getting a pizza over $16.5in$?

Getting Huge Pizza

Your local pizza shop claims that their large is at least 16in or it is free. Their pizza is normally distributed with $\mu = 16.3in$ and $\sigma = 0.2in$. What is the probability of getting a pizza between $15.95in$ and $16.63in$?

Normal Distribution

Recall that the only way we can analyze probabilities in the continuous case is with the CDF:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x-\mu)^2}{2\sigma^2}}$$

$$F(a) = \int_{-\infty}^{a} f(x)\, dx$$

$$\Pr(X < a) = F(a)$$

$$\Pr(X > a) = 1.0 - F(a)$$

$$\Pr(a < X < b) = F(b) - F(a)$$