Probability in Computing

CS 237

LECTURE 26

Last time

- Poisson process Poisson process
- Use of probability in algorithms (sorting uniform input in linear time)

Today

- Use of probability in data structures (hashing and Bloom filters)
- Sublinear-time algorithms

Reminder

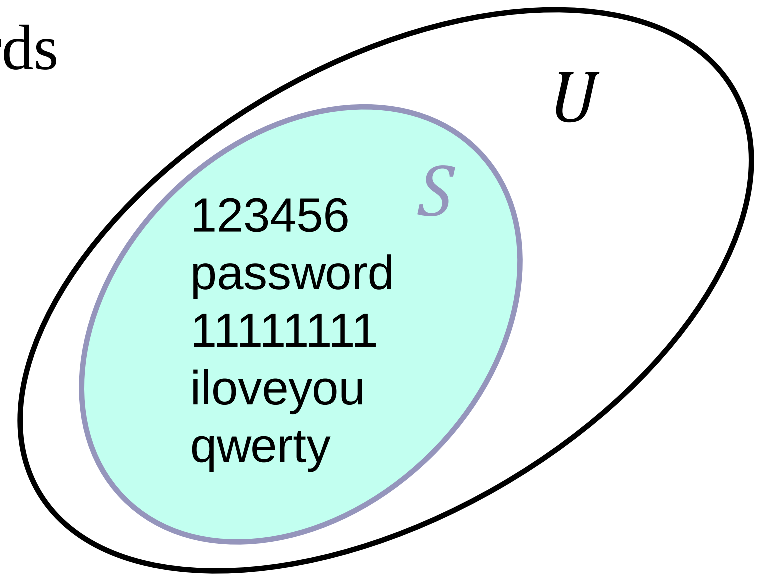
- HW 12 is due today

Static dictionary problem

Motivating example

Password checker to prevent people from using common passwords.

- S is the set of common passwords
- **Universe:** set U
- $S \subseteq U$ and $m = |S|$
- $m \ll |U|$



Goal: A data structure for storing $\boldsymbol{S}$ that supports the search query

"*Does* $w \in \boldsymbol{S}$?" for all words $w \in \boldsymbol{U}$.

Solutions

Deterministic solutions

- Store $\boldsymbol{S}$ as a sorted array (or as a binary search tree)

Search time: **O**($\log m$), **Space**: **O**(m)

- Store an array that for each $w \in \boldsymbol{U}$ has 1 if $w \in \boldsymbol{S}$ and 0 otherwise.

Search time: **O**(1), **Space**: **O**($|\boldsymbol{U}|$)

A randomized solution

- Hashing

Hashing with Chaining

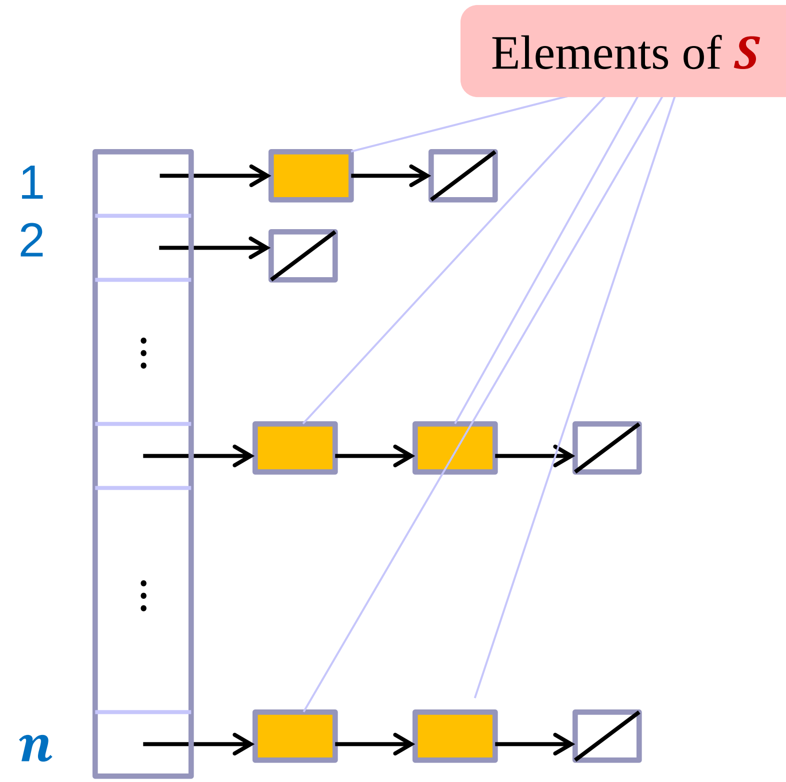
- **Hash table:** $\boldsymbol{n}$ bins, words that fall in the same bin are chained into a linked list.
- **Hash function:** $h : \mathrm{U} \rightarrow \{1, \ldots, n\}$

To construct the table

hash all elements of S

To search for word $\boldsymbol{w}$

check if w is in bin $h(w)$



A random hash function

- Simplifying assumption: hash function h is selected at random:

$$\Pr[h(w) = j] = \frac{1}{n} \text{ for all } w \in U \text{ and } j \in \{1, \dots, n\}$$

- Once h is chosen, every evaluation of h yields the same answer.

Search time:

- If $w \notin S$, expected number of words in bin $h(w)$ is
- If $w \in S$, expected number of words in bin $h(w)$ is

If we set $n = m$, then

- the expected search time is O(1)
- max time to search is max load: w.p. close to 1, it is $\Theta\left(\frac{\ln m}{\ln \ln m}\right)$

Faster than a search tree, with space still $\Theta(m)$.

Approximate solution

for static dictionary problem

- **False positives:** If $w \in \boldsymbol{S}$, our data structure must answer correctly. If $w \notin \boldsymbol{S}$, we may err with small probability.
- E.g., we prevent all unsuitable passwords and some suitable ones, too.

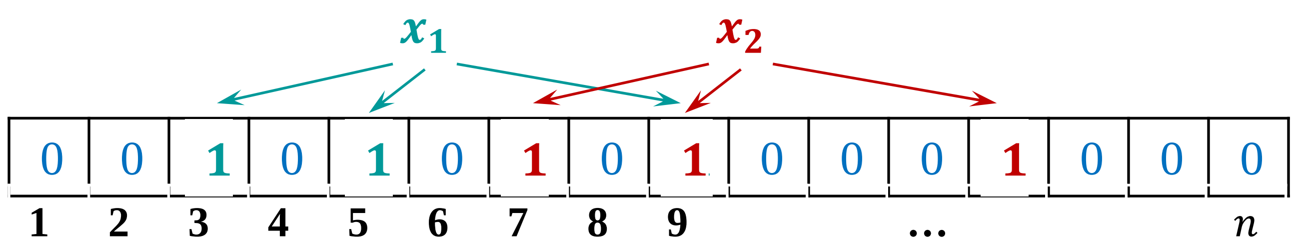
- Bloom filters
  - give trade off between space and false positive probability
  - have parameters k, n

Bloom filter with n bits and k hash functions

- Bloom filter: array of n bits $A[1], \ldots, A[n]$
  - Initially: all bits are 0

0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
1	**2**	**3**	**4**	**5**	**6**	**7**	**8**	**9**			**…**				n

  - k independent random hash functions $h_1, \ldots, h_k$ with range $\{1, \ldots, n\}$
- To represent set S
  - For each $x \in S$ and $i \in \{1, \ldots, k\}$, set bits $A[h_i(x)]$ to 1.

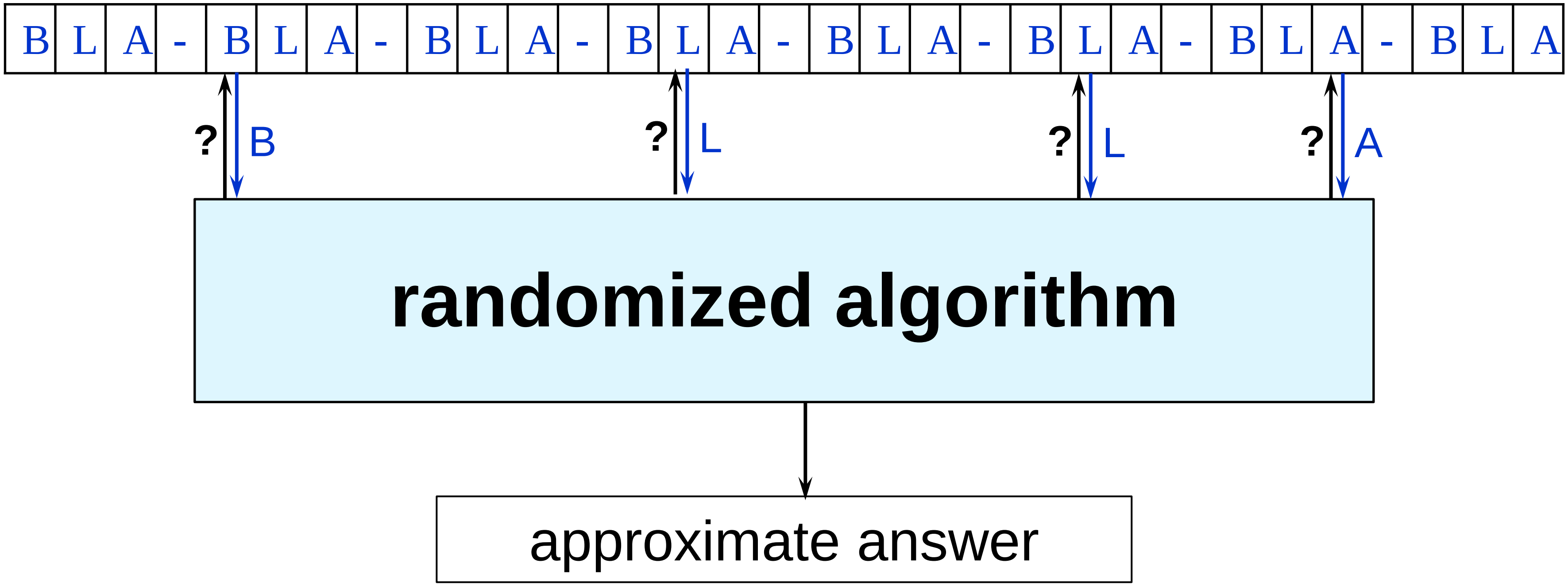


- To decide if $w \in S$:
  - If for all $i \in \{1, \ldots, k\}$, bits $A[h_i(w)] = 1$, accept, o.w. reject.

Analysis of False Positive rate

- For any n, we can set $k \approx \frac{n}{m} \ln 2$.
- Consider $w \in U - S$.
- Let $B_i = A[h_i(w)]$ for all $i \in \{1, \dots, k\}$
- After m elements hashed into Bloom filter, $\Pr[B_i = 0] =$

A Sublinear-Time Algorithm



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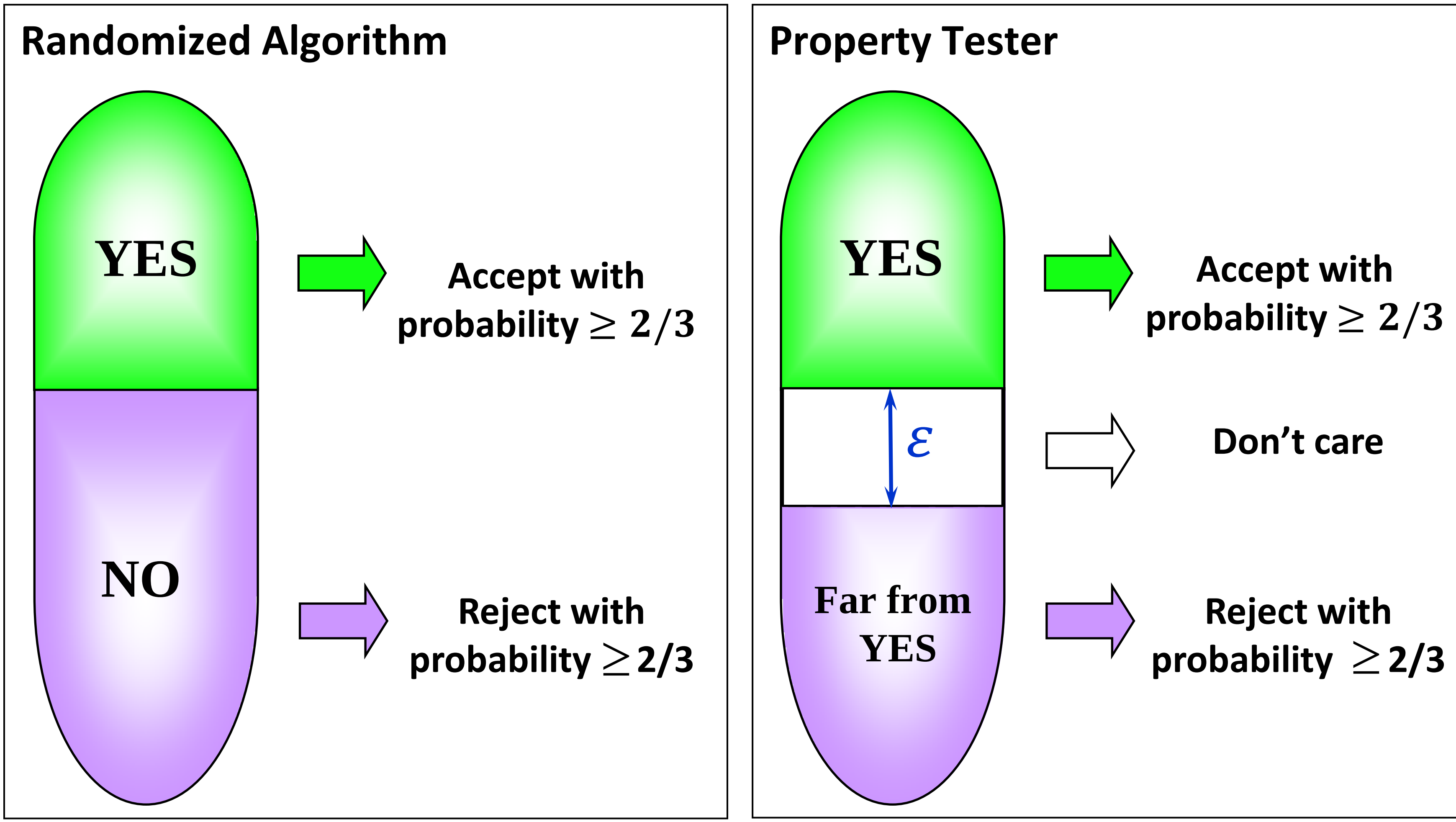
Randomized algorithms; property testers



ε-far = differs in many places ($\geq \varepsilon$ fraction of places)

Example: Testing if a List is Sorted

Input: a list of n numbers $x_1, x_2, \ldots, x_n$

- A list of numbers is sorted if $x_1 \leq x_2 \leq \cdots \leq x_n$.

- Question: Is the list sorted?

  Requires reading entire list: $\Omega(n)$ time

- Approximate version: Is the list sorted or ε-far from sorted?

  (An ε fraction of x_i 's have to be changed to make it sorted.)

  $O\left(\frac{\log n}{\varepsilon}\right)$ time

Sortedness Testing: Attempts

1. Test: Pick a uniformly random $i \in \{1, \dots, n-1\}$ and reject if $x_i > x_{i+1}$.

Fails on:

1	1	1	1	0	0	0	0

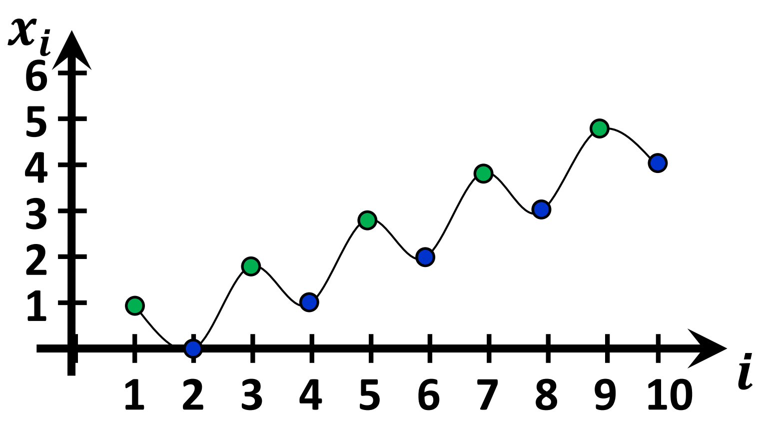
← 1/2-far from sorted

2. Test: Pick uniformly random $i < j$ in $\{1, \dots, n\}$ and reject if $x_i > x_j$.

Fails on:

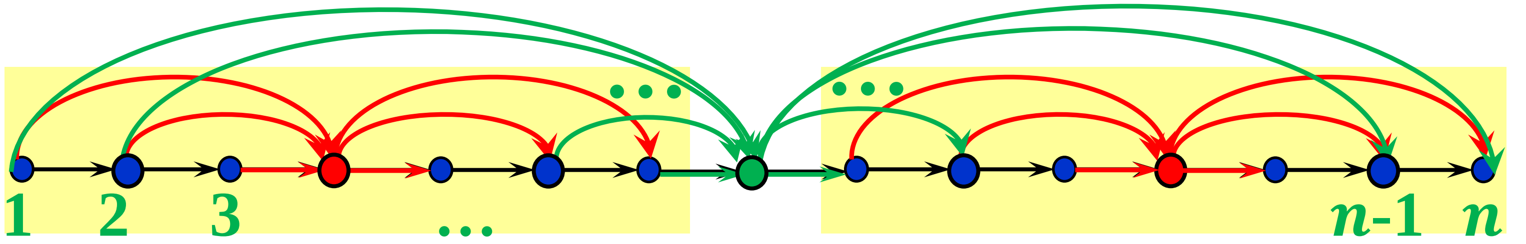
1	0	2	1	3	2	4	3	5	4

← 1/2-far from sorted



Is a list Sorted or ε-far from Sorted?

Idea: Associate positions in the list with vertices of the directed line.



Construct a graph (2-spanner)

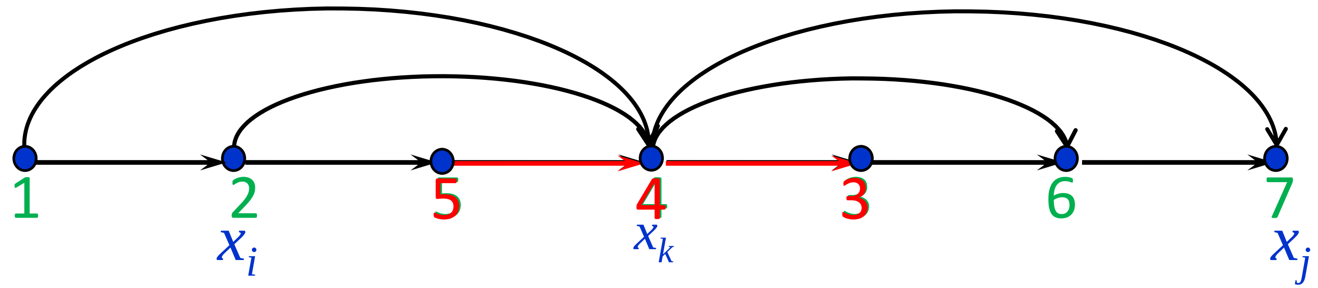
$\leq n \log n$ edges

- by adding a few "shortcut" edges (i, j) for $i < j$
- where each pair of vertices is connected by a path of length at most 2

Is a list Sorted or ε-far from Sorted?

Test

Pick a random edge (i, j) from the 2-spanner and **reject** if $x_i > x_j$.



Analysis:

- Call an edge (i, j) **violated** if $x_i > x_j$, and **satisfied** otherwise.
- If i is an endpoint of a **violated** edge, call x_i **bad**. Otherwise, call it **good**.

Claim 1. All **good** numbers are sorted.

Proof: Consider any two good numbers, x_i and x_j.

They are connected by a path of (at most) two **satisfied** edges $(i, k), (k, j)$

$\Rightarrow x_i \leq x_k$ and $x_k \leq x_j$

$$\Rightarrow \; x_i \leq x_j$$

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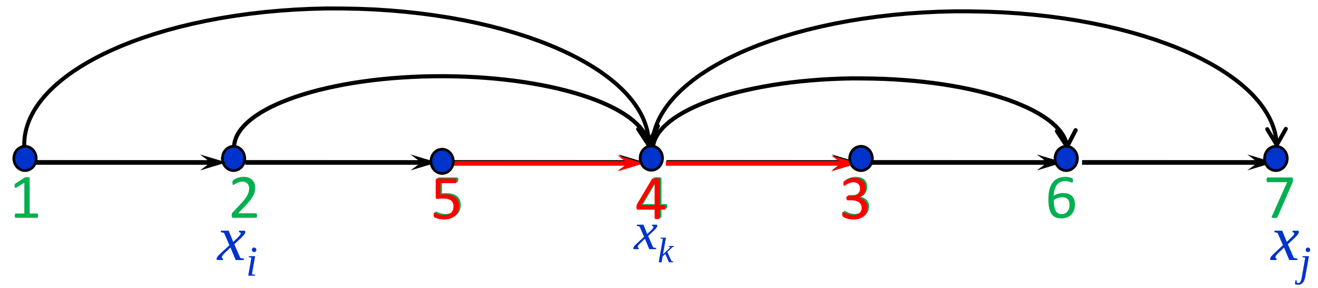

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Claim 1. All **good** numbers are sorted.

Claim 2. An ε-far list **violates** $\geq \varepsilon/(2\log n)$ fraction of edges in 2-spanner.

Proof: If a list is ε-far from sorted, it has $\geq \varepsilon n$ **bad** numbers. (Claim 1)

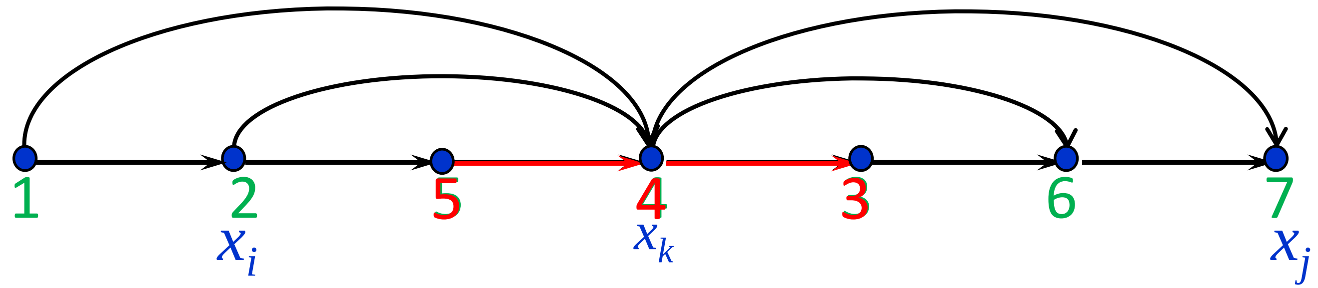
- Each **violated** edge contributes 2 **bad** numbers.

2-spanner has $\geq \frac{\varepsilon n}{2}$ **violated** edges out of $n \log n$.

Is a list Sorted or ε-far from Sorted?

Test

Pick a random edge (i, j) from the 2-spanner and **reject** if $x_i > x_j$.



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Algorithm

Sample $\frac{4 \log n}{\varepsilon}$ edges (i, j) from the 2-spanner and **reject** if $x_i > x_j$.

Guarantee: All sorted lists are accepted.

All lists that are ε-far from sorted are rejected with probability $\geq 2/3$.

Testing if a List is Sorted: Summary

We can determine if a list of n numbers is sorted or ε-far from sorted in $O\left(\frac{\log n}{\varepsilon}\right)$ time.

Sublinear Algorithms: Summary

- Many problems admit sublinear-time algorithms
- Algorithms are often simple
- Analysis requires creation of interesting combinatorial, geometric and algebraic tools