CS 237: *Randomness in Computing* Professors: Tiago Januario, Sofya Raskhodnikova
Boston University January 22, 2023

Homework 1 – Due Thursday, January 26, 2023 by midnight

- Provide step-by-step explanations, not just answers. Answers without explanations will earn a small fraction of points.
- Submit your solutions on Gradescope. Don't forget to include information about your collaborators (or say "Collaborators: none").

Problems

0. (**0 points**) The following steps are required to get you started in the course. Please complete them **before** your discussion section on Friday.

   (a) Make sure you are signed up on piazza at `https://piazza.com/bu/spring2023/cascs237/home` using your BU email address.

   (b) Sign up on Gradescope using your BU email address and the code **G2K3Y5**.

   (c) Register for Top Hat. If you already have an account, `https://app.tophat.com/e/033357` will take you to our course. Otherwise:

      i. Go to `https://app.tophat.com/register/student`

      ii. Click "Search by school" and input Boston University

      iii. Search for our course with the following join code: **033357**

   If a paid subscription is required, it will be listed at checkout when you enroll in our Top Hat course. Should you require assistance with Top Hat at any time please contact their Support Team directly by way of email (support@tophat.com), the in-app support button, or by calling 1-888-663-5491.

   (d) Read and sign the Collaboration and Honesty Policy and submit it on Gradescope. We will be able to grade your homework only after you complete this step.

   (e) (**Nameplate**) Please print out (or make by hand) a nameplate with your name and bring it to every lecture and discussion. A template is available at the bottom of the course web page.

   (f) Check out the following links and resources:

      i. course webpage: `https://cs-people.bu.edu/sofya/cs237/`;

      ii. course textbooks: `https://cs-people.bu.edu/aene/cs237fa21/mcs.pdf`
      H. Pishro-Nik, *Introduction to probability, statistics, and random processes,* available at `https://www.probabilitycourse.com`, Kappa Research LLC, 2014.

      iii. supplementary textbook to review proof techniques:
      Richard Hammack. *Book of Proof*: `http://www.people.vcu.edu/~rhammack/BookOfProof/`

   (g) Familiarize yourself with the homework template files at the bottom of the course webpage. Note that each problem must include a note about collaborators (even if you did the problem by yourself).

   (h) Follow instructions posted on the course web page to set up for Google Colaboratory: `https://cs-people.bu.edu/sofya/cs237/Organization_handouts/CS237___Instructions_for_setting_up_Google_Colab.pdf`

1. (**Counting, 10 points**) We start by giving you a solved problem, so that you understand the level of detail we are expecting in your solutions.

   Sample Problem: *In how many ways can you make a pair of "words" using all the letters of the word* randomness*? For the purposes of this problem, a "word" is any nonempty sequence of letters. For example, one way would be to make words* manndoess *and* r*. Note that reordering the two words results in a new pair.*

   Sample Solution: *First, we count the number of ways to arrange the 10 letters of* randomness *and then take into account the number of ways to break them down into two words. We number the letters from 1 to 10. There are 10! ways to permute these numbers. Notice that we have 8 unique letters and two pairs of letters. Switching the positions of the two letters* s *does not give us a new arrangement of the 10 letters. Since there are two ways to order their positions (9 and 10), we need to divide our count by 2. Similar reasoning applies to letters* n*. So, we need to divide by another 2. The total number of ways to arrange the letters of the word* randomness *is thus (10!)/4. Finally, for each arrangement of the letters, there are 9 ways to put a divider between the two words: it can come after any letter, except for the last one. Therefore, the final answer is* $2.25 \cdot (10!)$.

   (a) A student is taking a multiple-choice quiz with 4 questions. Each question has 6 choices, only one of which can be selected. How many ways are there to choose a complete set of answers?

   (b) The same question as in part (a), but with the restriction that all questions have different answers. Assume that each question has choices (a), (b), (c), (d), (e), and (f).

   (c) In how many ways can Yael arrange 13 different beads on a circular bracelet?

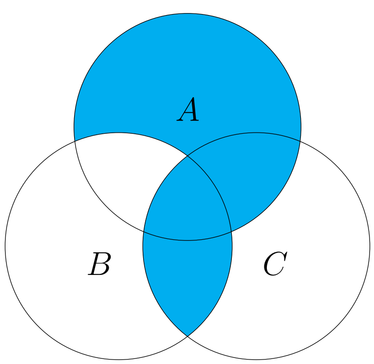
   (d) A yogurt shop has 6 different flavors and 10 different toppings. If a customer wants to get one flavor and two different toppings, how many choices does she have?

   (e) Let $n \geq 2$ be an integer. How many ways can n books be arranged on a shelf if two books must stay together?

2. (**Sets review, 10 points**)

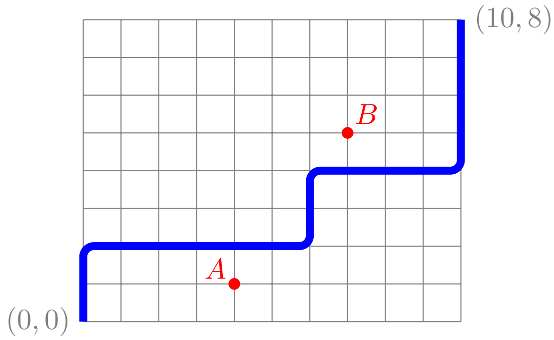
   (a) Find the number of sets S that satisfy $\{a, b\} \subseteq S \subseteq \{a, b, c, d, e, f\}$.

   (b) Use set notation to describe the set depicted on this Venn diagram.



   (c) Give a Venn diagram for $A \setminus (B \setminus C)$.

   (d) How many paths through the grid are there from $(0, 0)$ to $(10, 8)$ that avoid both points A and B? Such paths must have exactly 18 steps: 10 steps to the right and 8 steps up. One such path is shown in blue. *Hint:* Use the Inclusion-Exclusion Principle.



3. (**Proof techniques review, 10 points**)

   (a) (**Direct proof**) Show that for all integers n greater than 2, the number $1 + 2 + \cdots + n$ is composite.
   (Recall that a positive integer is *composite* if it has at least one divisor other than 1 and itself.)

   (b) (**Proof by contradiction**) Prove by contradiction that if $|x| + x > 0$, then x must be positive.

   (c) (**Induction**) Prove by induction that, for all integer $n \geq 2$,

$$\left(1 - \frac{1}{4}\right) \cdot \left(1 - \frac{1}{9}\right) \cdot \dots \cdot \left(1 - \frac{1}{n^2}\right) = \frac{n+1}{2n}.$$

4. (**Strong induction review, 10 points**) Suppose we are trying to divide a class of n students into groups of either 4 or 5 students.

   (a) Find an error in the following proof that a class with $n \geq 8$ students can be divided into groups of 4 or 5. That is, **identify the first incorrect sentence** and **explain** what went wrong. (*Note:* It is not enough to say that the statement is false. This only shows that there must be an error somewhere in the argument, but it is different from actually finding an error.)

   Proof. The proof is by strong induction. Let $P(n)$ be the proposition that a class with n students can be divided into teams of 4 or 5.

   Base case: We prove that $P(n)$ is true for $n = 8, 9$, and 10 by showing how to break classes of these sizes into groups of 4 or 5 students:

$$\begin{aligned} 8 &= 4 + 4; \\ 9 &= 4 + 5; \\ 10 &= 5 + 5. \end{aligned}$$

   Induction hypothesis: Next, we must show that $P(8), \ldots, P(n)$ imply $P(n+1)$ for all $n \geq 10$. That is, we assume that $P(8), \ldots, P(n)$ are all true and show how to divide up a class of $n+1$ students into groups of 4 or 5. We first form one group of 4 students. Then we can divide the remaining $n - 3$ students into groups of 4 or 5 by the assumption $P(n-3)$. This proves $P(n+1)$, and so the claim holds by induction. □

   (b) Provide a correct strong induction proof that a class with $n \geq 12$ students can be divided into groups of 4 or 5.

5. (**Calculus review, 10 points**)

   (a) Solve the following equations:

   i. $\log_2(x) + \log_2(3+x) = 2$
   ii. $5e^{0.12x} = 10e^{0.08x}$

(b) Find the inverse of the function $f(x) = 2e^{3x-2} + 1$.

(c) Find the derivatives of:

   i. $f(x) = 2x^3 - 2x^{-3} + 6x$
   ii. $f(x) = (2x^3 - 2x^{-3} + 6x)^{-2}$

(d) Evaluate $\lim_{x \to 9} \dfrac{x-9}{\sqrt{x}-3}$.

(e) Evaluate the integral $\displaystyle\int_0^2 e^x - 3x^2 + 1\, dx$.

6. (**Programming assignment**) The programming problems are posted in a file with extension .ipynb under Resourses in Piazza.