Homework 7 – Due Friday, November 2, 2018 before noon

Reminder  Collaboration is permitted, but you must write the solutions by yourself without assistance, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises  Please practice on exercises and solved problems in Chapter 4 as well as on the exercises below. Do not hand them in. The material they cover may appear on exams.

1. Please practice on exercises and solved problems in Chapter 5.

2. (True/False and Justify) Decide whether each statement is true or false and briefly justify your answer.

T  F  There exists an enumerator $\text{TM}$ that prints a set $S$ of $\text{TM}$ descriptions, such that $S$ includes descriptions of $\text{TMs}$ that decide infinitely many different languages.

T  F  There exists an enumerator $\text{TM}$ that prints a set $S$ of $\text{TM}$ descriptions, such that $S$ includes a description of at least one $\text{TM}$ for each language over alphabet $\{0, 1\}$.

T  F  There exists an enumerator $\text{TM}$ that prints the set $S$ of $\text{TM}$ descriptions that consists of descriptions of all $\text{TMs}$ whose language is empty.

T  F  If $A$ is a decidable language and $B$ is a Turing-recognizable language, then $A \setminus B$ must be Turing-recognizable.

T  F  A two-dimensional Turing machine is like an ordinary Turing machine except that its tape storage consists of a two-dimensional tape, corresponding to the upper right quadrant of the plane. Each tape cell is a unit square. In one step, the single tape head can move left, right, up or down. A two-dimensional $\text{TM}$ starts with its input written on consecutive cells, starting from the lowest leftmost cell (where the head is located) and going right. The class of languages recognized by two-dimensional Turing machines is exactly the Turing-recognizable languages.

Problems

1. (Undecidable languages) For each of the parts, formulate the given problem as a language and prove it is undecidable.

   (a) (Smooth $\text{TM}$) Let $\Sigma = \{0, 1, \ldots, 9\}$. A string $w_1w_2\ldots w_n$, where $w_1, w_2, \ldots, w_n \in \Sigma$, is smooth if, for each position $i \in \{1, \ldots, n-1\}$ in the string, $|w_{i+1} - w_i| \leq 1$. In other words, as we read the string, each subsequent digit is the same as the previous one or differs from it by 1. For example, 5654321210 is smooth, but 576 is not. Observe that $\varepsilon$ and 1-digit strings are smooth, because they do not violate the smoothness requirement.

   A $\text{TM}$ is smoothness-obsessed if it accepts a string if and only if it is smooth. The problem is to determine whether a given $\text{TM}$ is a smoothness-obsessed.
(b) **Adding TM** A TM correctly adds if, given two binary numbers, separated by #, it halts with their sum (in binary) on its tape. (It does not matter what it does on other inputs.) Consider the problem of determining whether a TM correctly adds. Formulate this problem as a language and prove it is undecidable.

2. **(Enthusiastic TM)** Consider the problem of determining whether a given TM ever writes “332” on three adjacent squares of its tape. You may assume that the input alphabet of this TM is \{0, 1\} and the tape alphabet is \{0, 1, 2, \ldots, 9\}.

   (a) Formulate this problem as a language \textsc{Enthusiastic}_{TM}.

   (b) Show \textsc{Enthusiastic}_{TM} is undecidable.

   (c) Prove that \textsc{Enthusiastic}_{TM} is Turing-recognizable.

   (d) Is \textsc{Enthusiastic}_{TM} Turing-recognizable? Prove or disprove.

3. **(Recognizable and unrecognizable languages, 15 points)**

   (a) **(Complement of ALLCFG)** Give a definition of \textsc{ALLCFG}, analogous to \textsc{ALLDFA} in Problem 4.3 in the book. Prove that the complement of \textsc{ALLCFG} is Turing-recognizable.

   (b) **(DECIDER_{TM})** Let \textsc{Decider}_{TM} = \{\langle M \rangle \mid M \text{ is a TM that halts on every input}\}. Prove the following statements.

      i. \textsc{Decider}_{TM} is not Turing-recognizable (i.e., \textsc{Decider}_{TM} is not co-Turing-recognizable).

      ii. \textsc{Decider}_{TM} is not Turing-recognizable.

4* **(Optional, no collaboration is allowed)** Let \(A\) be a Turing-recognizable language consisting of descriptions of Turing machines, \{\langle M_1 \rangle, \langle M_2 \rangle, \ldots \}, where every \(M_i\) is a decider. Prove that some decidable language \(D\) is not decided by any decider \(M_i\) whose description appears in \(A\). (*Hint: Use diagonalization and consider an enumerator for \(A\).*)

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\(^1\)on some input