Intro to Theory of Computation

Lecture 5

Last time:
• Closure properties.
• Equivalence of NFAs, DFAs and regular expressions

Today:
• Proving that a language is not regular: pumping lemma
Conversion procedures

DFA ⇌ NFA

Regular Language ⇌ Regular Expression

definition
Design an NFA for the language:

\[ \{0^n1^n \mid 0 < n \leq 2\} \]

\[ \{0^n1^n \mid 0 < n \leq k\} \]

\[ \{0^n1^n \mid n > 0\} \]

(For R a regexp, \( R^2 \) means \( RR \), and \( R^n \) means \( RR \ldots R \))
SOME LANGUAGES ARE NOT REGULAR!

\[ B = \{0^n1^n \mid n \geq 0\} \text{ is NOT regular!} \]
Let M be a k-state DFA that recognizes B.

Consider the path M takes on $0^k1^k$:

$$q_0q_1q_2... q_iq_{i+1} q_j q_k ... q_{2k} \in F$$

0000...00..0..011111...11

There must be $i < j \leq k$ such that $q_i = q_j$

M accepts $0^{k-(j-i)}1^k \notin B$!

So M does not recognize the language B.
REGULAR OR NOT?

\[ C = \{ w \mid w \text{ has equal number of 1s and 0s} \} \]

NOT REGULAR

\[ D = \{ w \mid w \text{ has equal number of occurrences of 01 and 10} \} \]

\[ (0\Sigma^*0) \cup (1\Sigma^*1) \cup 1 \cup 0 \cup \varepsilon \]
THE PUMPING LEMMA

Let \( L \) be a regular language with \( |L| = \infty \)

Then there exists a length \( p \) such that

if \( w \in L \) and \( |w| \geq p \) then

\( w \) can be split into three parts \( w=xyz \) where:

1. \( |y| > 0 \)
2. \( |xy| \leq p \)
3. \( xy^iz \in L \) for all \( i \geq 0 \)
THE PUMPING LEMMA

Example:

Let \( L = 0^*1^* \); \( p = 1 \)

\( w = 011 \)

\( x = \varepsilon \)

\( y = 0 \)

\( z = 11 \)

If \( w \in L \) and \( |w| \geq p \) then \( w = xyz \), where:

1. \( |y| > 0 \)
2. \( |xy| \leq p \)
3. \( xy^iz \in L \) for all \( i \geq 0 \)

Let \( L = (0 \cup 1)^2^* \); \( p = 2 \)

\( w = 12 \)

\( x = 1 \)

\( y = 2 \)

\( z = \varepsilon \)
Let $M$ be a DFA that recognizes $L$. Let $p$ be the number of states in $M$. Assume $w \in L$ is such that $|w| \geq p$.

We show $w = xyz$

1. $|y| > 0$
2. $|xy| \leq p$
3. $xy^iz \in L$ for all $i \geq 0$

There must be $j > i$ such that $j \leq p$ and $q_i = q_j$. 

Sofya Raskhodnikova; based on slides by Nick Hopper
Use the pumping lemma to prove that \( B = \{0^n1^n \mid n \geq 0\} \) is not regular

**Hint:** Assume \( B \) is regular. Let \( p \) be the pumping length.

Try pumping \( w = 0^p1^p \).

If \( B \) is regular, \( w \) can be split into \( w = xyz \), where

1. \( |y| > 0 \)
2. \( |xy| \leq p \)
3. \( xy^iz \in B \) for all \( i \geq 0 \)

\( y \) is all 0s: \( xyyz \) has more 0s than 1s

**Contradiction!**
Proof by contradiction: assume $L$ is regular.

Then there is a pumping length $p$.

Find a string $w \in L$ with $|w| \geq p$.

Show that no matter how you choose $xyz$, $w$ cannot be pumped!

Conclude that $L$ is not regular.
Pumping lemma as a game

1. **YOU** pick the language $L$ to be proved nonregular.

2. **ADVERSARY** picks $p$, but doesn't reveal to **YOU** what $p$ is; **YOU** must devise a play for all possible $p$'s.

3. **YOU** pick $w$, which should depend on $p$ and which must be of length at least $p$.

4. **ADVERSARY** divides $w$ into $x, y, z$, obeying conditions stipulated in the pumping lemma: $|y| > 0$ and $|xy| \leq p$. Again, **ADVERSARY** does not tell **YOU** what $x, y, z$ are, although they must obey the constraints.

5. **YOU** win by picking $i$, which may be a function of $p, x, y, z$, such that $xy^i z$ is not in $L$. 
**EVENPALINDROMES** = \{ ww^R \mid w \in \{0,1\}^* \} is not regular.

Proof: Assume … pumping length \( p \)

Find a \( w \in EVENPALINDROMES \) longer than \( p \)

\( 0^p1^p1^p0^p \)

Show that \( w \) cannot be pumped:

\[
\begin{align*}
  w &= 00\ldots0011\ldots1100\ldots00 \\
  xyyz &= 00\ldots00011\ldots1100\ldots00
\end{align*}
\]

\( y \) must be in this part

\[
\begin{align*}
  \underbrace{00\ldots00}_p & \quad \underbrace{11\ldots11}_2p \quad \underbrace{00\ldots00}_p \\
  \underbrace{y}_p & > p \quad \underbrace{2p}_2p \quad \underbrace{y}_p
\end{align*}
\]
**EVENPALINDROMES** = \{ ww^R \mid w \in \{0,1\}^* \} is not regular.

**Proof:** Assume ... pumping length \( p \)

Find a \( w \in EVENPALINDROMES \) longer than \( p \)

Show that \( w \) cannot be pumped:

If \( w = xyz \) with \( |xy| \leq p \) then
\( y = 0^J \) for some \( J>0 \).

Then \( xyyz = 0^{P+J}1^{2P}0^P \notin EVENPALINDROMES \)

**Contradiction!**
Prove $C = \{ 0^i1^j \mid i > j \geq 0 \}$ is not regular.

Proof: Assume ... pumping length $p$

What string can you choose in your next move?

A. 00011

B. $0^p1^p$

C. $0^{p/2}1^{p/2-1}$

D. $0^{p+1}1^p$

E. $0^{p+2}1^p$

F. More than one choice above works.
Prove $C = \{0^i1^j \mid i > j \geq 0\}$ is not regular.

Proof: Assume ... pumping length $p$

Find a $w \in C$ of length at least $p$

$0^{p+1}1^p$

Show that $w$ cannot be pumped:

$w = 00...0011...11$

$y$ must be in this part

$xyyz = 00...00011...11 \quad > p+1$

$xz = 0...0011...11 \quad \leq p$
Prove \( C = \{ 0^i 1^j \mid i > j \geq 0 \} \) is not regular.

Proof: Assume ... pumping length p

Find a \( w \in C \) of length at least p

\[ 0^{p+1} 1^p \]

If \( w = xyz \) with \( |xy| \geq p \) then

\( y = 0^j \) for some \( J \geq 1 \).

Then \( xy^0z = xz = 0^{p+1-J} 1^p \notin C \)

Contradiction!
Let $\text{BALANCED} = \{ w \mid w \text{ has an equal # of 1s and 0s} \}$

Assume ... there is a $p$

Find a $w \in \text{BALANCED}$ of length at least $p$

Show that $w$ cannot be pumped:

If $w = xyz$ with $|xy| \leq p$ then

$y = 0^J$ for some $J > 0$.

Then $x y y z = 0^{p+J} 1^p \not\in \text{BALANCED}$
Pumping a language can be lots of work…
Let’s try to reuse that work!

\[ \{0^n1^n \mid n \geq 0\} = \text{BALANCED} \cap 0^*1^* \]

If BALANCED is regular then so is \{0^n1^n \mid n \geq 0\}
Prove: A is not regular

Proof by contradiction: Suppose A is regular. Then $A \cap C (= B)$ is regular. But B is not regular. A contradiction.
Prove $A = \{0^i1^j \mid i \neq j\}$ is not regular using $B = \{0^n1^n \mid n \geq 0\}$

$\neg A = B \cup \{ \text{strings that mix 0s and 1s} \}$

$\neg A \cap 0^*1^* = B$
Let $F = \{a^ib^jc^k \mid i, j, k \geq 0 \text{ and } i=1 \Rightarrow j=k\}$

$F$ has pumping length 2!

- $i = 0$
- $i = 1$
- $i = 2$
- $i > 2$

HW2: prove that $F$ is not regular.

A non-regular language may still satisfy the pumping lemma.