LECTURE 16

Last time
• $A_{TM}$ is unrecognizable
• Reductions

Today
• Reductions
• Mapping reductions

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3/17/2016
Problems in language theory

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We want to prove that language \( L \) is undecidable.

**Idea:** Use a proof by contradiction.

1. Suppose to the contrary that \( L \) is decidable.
2. Use a decider for \( L \) as a subroutine to construct a decider for \( A_{TM} \).
3. But \( A_{TM} \) is undecidable. Contradiction!
Exercise

To prove that $E_{TM}$ is undecidable

A. we assumed $E_{TM}$ had a decider and used it to construct a decider for $A_{TM}$

B. we assumed $A_{TM}$ had a decider and used it to construct a decider for $E_{TM}$

C. we constructed a TM $S$ that on input $<M, w>$ decides whether $M$ accepts $w$, assuming the existence of a TM $R$ that decides on input $<M'>$ whether the language of $<M'>$ is empty

D. There is more than one correct answer.

E. None of the above.
Prove that \( \text{EQ}_{\text{TM}} \) is undecidable

\[ \text{EQ}_{\text{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2) \} \]

Proof: Suppose to the contrary that \( \text{EQ}_{\text{TM}} \) is decidable, and let \( R \) be a TM that decides it.

We construct TM \( S \) that decides \( A_{\text{TM}} \).

\( S = \text{``On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:} \)

1. Construct TMs \( M', M'' \).
   \[
   M' = \text{``On input } x, \text{ 1. Ignore the input. 2. Run TM } M \text{ on input } w. \text{ 3. If it accepts, accept.}”
   \]
   \[
   M'' = \text{``Accept.”}
   \]
2. Run TM \( R \) on input \( \langle M', M'' \rangle \).
3. If it accepts, accept. O.w. reject.”
Proof 2 that $\text{EQ}_{\text{TM}}$ is undecidable

$\text{EQ}_{\text{TM}} = \{\langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2)\}$

Proof: Suppose to the contrary that $\text{EQ}_{\text{TM}}$ is decidable, and let $R$ be a TM that decides it.

We construct TM $S$ that decides $\text{E}_{\text{TM}}$. What do we change?

$S = \text{``On input } \langle M \rangle \text{, where } M \text{ is a TM and } w \text{ is a string:} $

1. Construct TM $M'$.
   
   $M' = \text{``Reject.''}$
   
   1. Ignore the input.
   2. Run TM $M$ on input $w$.
   3. If it accepts, accept.''

2. Run TM $R$ on input $\langle M, M' \rangle$.

3. If it accepts, accept. O.w. reject.''

$L_{M'}$
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Proving undecidability and unrecognizability

Mapping Reductions
A function $f : \Sigma^* \rightarrow \Sigma^*$ is **computable** if some TM $M$, on every input $w$, halts with only $f(w)$ on its tape.

**Example 1:** $f(\langle x, y \rangle) = x + y$.

**Example 2:** $f(\langle M, w \rangle) = \langle M' \rangle$, where $M$ is a TM and $w$ is a string, and $M'$ is a TM that ignores its input and runs $M$ on $w$. 
Given languages $A$ and $B,$

$$A \leq^m B$$

if there is a computable function $f,$

such that for all strings $w,$

$$w \in A \text{ iff } f(w) \in B.$$
Exercise

If $\bar{A} \leq_m \bar{B}$, we can conclude that

A. $A \leq_m B$
B. $B \leq_m A$
C. $\bar{A} \leq_m B$
D. $\bar{B} \leq_m A$
E. None of the above.
**Theorem.** If $A \leq_m B$ and $B$ is decidable, then $A$ is decidable.

**Proof:** Let $M$ be a decider for $B$ and $f$ be a mapping reduction from $A$ to $B$. Construct a decider for $A$:
```
`On input w:
1. Compute f(w).
2. Run M on f(w).
3. If it accepts, accept. O.w. reject.```

Theorem. If $A \leq_m B$ and $B$ is decidable, then $A$ is decidable.

Corollary. If $A \leq_m B$ and $A$ is undecidable, then $B$ is undecidable.

Example: If $A_{TM} \leq_m B$, then $B$ is undecidable.
Theorem. If $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.

Proof: Let $M$ be a TM that recognizes $B$ and $f$ be a mapping reduction from $A$ to $B$. Construct a TM that recognizes $A$:
``On input $w$:
1. Compute $f(w)$.
2. Run $M$ on $f(w)$.
3. If it accepts, accept. O.w. reject.\"
Using mapping reductions to prove unrecognizability

**Theorem.** If $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.

**Corollary.** If $A \leq_m B$ and $A$ is unrecognizable, then $B$ is unrecognizable.

**Example:** If $A_{TM} \leq_m B$, then $B$ is unrecognizable.
Old proof that $EQ_{TM}$ is undecidable

$EQ_{TM} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2) \}$

**Proof:** Suppose to the contrary that $EQ_{TM}$ is decidable, and let $R$ be a TM that decides it. We construct TM $S$ that decides $A_{TM}$.

$S =$ \`` On input $\langle M, w \rangle$, where $M$ is a TM and $w$ is a string:


$M' =$ \`` On input $x$, 
1. Ignore the input.
2. Run TM $M$ on input $w$.
3. If it accepts, accept."

$M'' =$ \`` Accept.""

2. Run TM $R$ on input $\langle M', M'' \rangle$.

3. If it accepts, accept. O.w. reject.""
Proof: The following TM computes the reduction:

F = “On input \( \langle M, w \rangle \), where \( M \) is a TM and \( w \) is a string:

1. Construct TMs \( M', M'' \).
   
   \( M' = \) “On input \( x \),
   
   1. Ignore the input.
   2. Run TM \( M \) on input \( w \).
   3. If it accepts, accept.”

   \( M'' = \) “Accept.”

2. Output \( \langle M', M'' \rangle \).”
Conclusions from $A_{TM} \leq_m EQ_{TM}$

1. Since $A_{TM}$ is undecidable, so is $EQ_{TM}$

2. $A_{TM} \leq_m EQ_{TM}$
   Since $A_{TM}$ is unrecognizable, so is $EQ_{TM}$
Prove that $\text{EQ}_{TM}$ is unrecognizable

Proof: We give a mapping reduction $\overline{A_{TM}} \leq_m \text{EQ}_{TM}$

The following TM computes the reduction:

$F = \text{"On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:}\$

   
   $M' = \text{"On input } x, \$
   1. Ignore the input.
   2. Run TM $M$ on input $w$.
   3. If it accepts, accept.”

   $M'' = \text{"Reject.”}$

2. Output $\langle M', M'' \rangle$.”
# Problems in language theory

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