Last time
• $A_{TM}$ is unrecognizable
• Reductions

Today
• Reductions
• Mapping reductions

Sofya Raskhodnikova
Problems in language theory

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We want to prove that language $L$ is undecidable.

**Idea:** Use a proof by contradiction.

1. Suppose to the contrary that $L$ is decidable.
2. Use a decider for $L$ as a subroutine to construct a decider for $A_{TM}$.
3. But $A_{TM}$ is undecidable. Contradiction!
Exercise

To prove that $E_{TM}$ is undecidable

A. we assumed $E_{TM}$ had a decider and used it to construct a decider for $A_{TM}$

B. we assumed $A_{TM}$ had a decider and used it to construct a decider for $E_{TM}$

C. we constructed a TM $S$ that on input $<M, w>$ decides whether $M$ accepts $w$, assuming the existence of a TM $R$ that decides on input $<M'>$ whether the language of $<M'>$ is empty

D. There is more than one correct answer.

E. None of the above.
Prove that $\text{EQ}_{\text{TM}}$ is undecidable

$\text{EQ}_{\text{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2) \}$

Proof: Suppose to the contrary that $\text{EQ}_{\text{TM}}$ is decidable, and let $R$ be a TM that decides it. We construct TM $S$ that decides $A_{\text{TM}}$.

$S = \text{``On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:} \text{''}$


$M' = \text{``On input } x, \text{''}$
1. Ignore the input.
2. Run TM $M$ on input $w$.
3. If it accepts, accept.”

$M'' = \text{``Accept.”}$$

2. Run TM $R$ on input $<M', M''>$. 

3. If it accepts, accept. O.w. reject.”
Proof 2 that \( \text{EQ}_{\text{TM}} \) is undecidable

\[ \text{EQ}_{\text{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2) \} \]

Proof: Suppose to the contrary that \( \text{EQ}_{\text{TM}} \) is decidable, and let \( R \) be a TM that decides it. We construct TM \( S \) that decides \( E_{\text{TM}} \). What do we change?

\[ S = \text{``On input } \langle M, w \rangle \text{, where } M \text{ is a TM and } w \text{ is a string:} \]

1. Construct TM \( M' \).

\[ M' = \text{``Reject.''} \]
   1. Ignore the input.
   2. Run TM \( M \) on input \( w \).
   3. If it accepts, accept.

2. Run TM \( R \) on input \( \langle M, M' \rangle \).

3. If it accepts, accept. O.w. reject.
## Problems in language theory

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Proving undecidability and unrecognizability

Mapping Reductions
A function $f : \Sigma^* \rightarrow \Sigma^*$ is **computable** if some TM $M$, on every input $w$, halts with only $f(w)$ on its tape.

**Example 1:** $f(\langle x, y \rangle) = x + y$.

**Example 2:** $f(\langle M, w \rangle) = \langle M' \rangle$, where $M$ is a TM and $w$ is a string, and $M'$ is a TM that ignore its input and runs $M$ on $w$. 
Given languages A and B,

\[ A \leq_m B \]

if there is a computable function \( f \), such that for all strings \( w \),

\[ w \in A \text{ iff } f(w) \in B. \]
If $\bar{A} \leq_m \bar{B}$, we can conclude that

A. $A \leq_m B$
B. $B \leq_m A$
C. $\bar{A} \leq_m B$
D. $\bar{B} \leq_m A$
E. None of the above.
**Theorem.** If $A \leq_m B$ and $B$ is decidable, then $A$ is decidable.

**Proof:** Let $M$ be a decider for $B$ and $f$ be a mapping reduction from $A$ to $B$. Construct a decider for $A$:

```
``\`On input $w$:
1. Compute $f(w)$.
2. Run $M$ on $f(w)$.
3. If it accepts, **accept**. O.w. **reject**.
```
Using mapping reductions to prove **undecidability**

**Theorem.** If $A \leq_m B$ and $B$ is decidable, then $A$ is decidable.

**Corollary.** If $A \leq_m B$ and $A$ is **undecidable**, then $B$ is **undecidable**.

**Example:** If $A_{TM} \leq_m B$, then $B$ is **undecidable**.
Theorem. If $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.

Proof: Let $M$ be a TM that recognizes $B$ and $f$ be a mapping reduction from $A$ to $B$. Construct a TM that recognizes $A$:
```
On input $w$:
1. Compute $f(w)$.
2. Run $M$ on $f(w)$.
3. If it accepts, accept. O.w. reject."
```
Using mapping reductions to prove unrecognizability

**Theorem.** If $A \leq_m B$ and $B$ is Turing-recognizable, then $A$ is Turing-recognizable.

**Corollary.** If $A \leq_m B$ and $A$ is unrecognizable, then $B$ is unrecognizable.

**Example:** If $\overline{A_{TM}} \leq_m B$, then $B$ is unrecognizable.
Old proof that $\text{EQ}_{\text{TM}}$ is undecidable

$\text{EQ}_{\text{TM}} = \{ \langle M_1, M_2 \rangle \mid M_1, M_2 \text{ are TMs, } L(M_1) = L(M_2) \}$

**Proof:** Suppose to the contrary that $\text{EQ}_{\text{TM}}$ is decidable, and let $R$ be a TM that decides it.

We construct TM $S$ that decides $A_{\text{TM}}$.

$S = \text{`` On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:}''$

   - $M' = \text{`` On input } x, 
   1. \text{ Ignore the input.}
   2. \text{ Run TM } M \text{ on input } w.$
   3. \text{ If it accepts, accept.''}$
   - $M'' = \text{`` Accept.''}$

2. Run TM $R$ on input $\langle M', M'' \rangle$.

3. If it accepts, accept. O.w. reject.''}
Proof: The following TM computes the reduction:

\[ F = \text{``On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:} \]

1. Construct TMs \( M', M'' \).
   \[ M' = \text{``On input } x, \]
   1. Ignore the input.
   2. Run TM \( M \) on input \( w \).
   3. If it accepts, accept.”

2. Output \( <M', M''> \).”

\[ A_{\text{TM}} \leq^m \text{EQ}_{\text{TM}} \]
Conclusions from $A_{TM} \leq_m EQ_{TM}$

1. Since $A_{TM}$ is undecidable, so is $EQ_{TM}$

2. $A_{TM} \leq_m EQ_{TM}$
   Since $A_{TM}$ is unrecognizable, so is $EQ_{TM}$
Prove that $EQ_{TM}$ is unrecognizable

Proof: We give a mapping reduction $A_{TM} \leq_m EQ_{TM}$

The following TM computes the reduction:

$F = \text{``On input } \langle M, w \rangle, \text{ where } M \text{ is a TM and } w \text{ is a string:}$$

   $M' = \text{``On input } x,$
   1. Ignore the input.
   2. Run TM $M$ on input $w$.
   3. If it accepts, accept.''

2. Output $\langle M', M'' \rangle$.

$A_{TM} \xrightarrow{f} EQ_{TM}$
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