Intro to Theory of Computation

CS 332

LECTURE 23

Last time

- Dynamic programming proof that all CFLs are in P
- Class **NP**

Today

- The **P** vs. **NP** question
- Polynomial-time reductions
- **NP**-completeness

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CS 332 Classes P, NP, EXP

- **P**. **Class of languages for which there is a poly-time algorithm.**

  algorithm that runs in time $O(n^k)$ for some k
- **EXP**. **Class of languages for which there is an exponential-time algorithm.**

  algorithm that runs in time $O(2^{n^k})$ for some k
- **NP**. **Class of languages for which there is a poly-time verifier.**

- **Lemma.** $P \subseteq NP$.
- **Lemma.** $NP \subseteq EXP$.
- **Lemma.** A language L is in NP iff L can be decided by a polynomial-time nondeterministic TM.

Exercise

To prove NP $\subseteq$ EXP, consider a language L$\in$ NP.

Then L has a verifier V that runs in time n^k.

We can construct an O(2^{n^k})-time TM for L as follows:

A. ``On input $\langle w, c\rangle$, where c is a certificate, run V on $\langle w, c\rangle$."

B. ``On input w, run V on $\langle w, c\rangle$ for all possible certificates c."

C. ``On input w, run V on $\langle w, c\rangle$ for all possible certificates c of length at most $|w|^k$."

D. ``On input w, run L on $\langle w, c\rangle$ for all possible certificates c of length at most $n^{|w|}$."

E. None of the above.

Nondeterministic time complexity classes

NTIME($f(n)$) is a class of languages.

$A \in$ **NTIME($f(n)$)** means that

some nondeterministic TM M

that runs in time O($f(n)$) decides A.

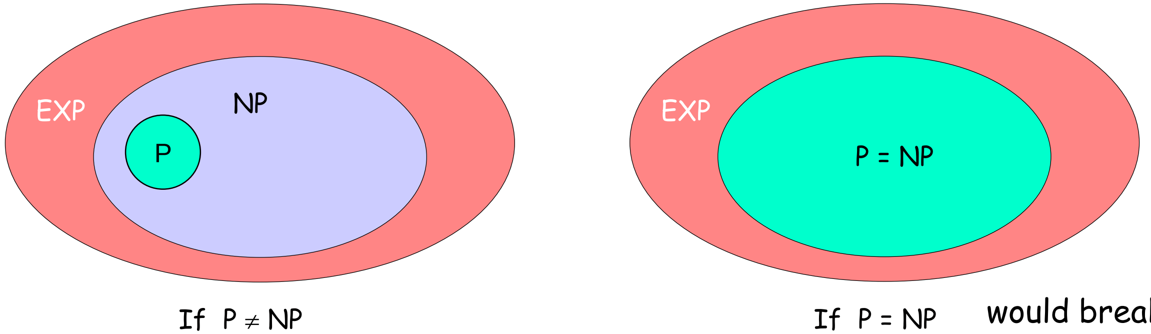
The class NP: alternative definition

NP is the class of languages decidable in polynomial time on a ***nondeterministic*** TM:

$$\boldsymbol{NP} = \bigcup_k NTIME(n^k).$$

P vs. NP

- Does P = NP? [Cook 1971, Edmonds, Levin, Yablonski, Gödel]
  - Is the decision problem as easy as the verification problem?
  - Clay $1 million prize.



would break RSA cryptography
(and potentially collapse economy)

- If yes: Efficient algorithms for UHamPath, SAT, TSP, factoring
- If no: No efficient algorithms possible for these problems.

- Consensus opinion on P = NP? Probably no.

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Classifying Problems

- **Desiderata:** classify problems according to those that can be solved in polynomial-time and those that cannot.
- Some problems *provably require exponential time* (Chapter 9):
  - Given a Turing machine, does it halt in at most *k* steps?
  - Given a board position in an *n*-by-*n* generalization of chess, can black guarantee a win?
- **Frustrating news:** huge number of fundamental problems have defied classification for decades.
- **Chapters 7.4-7.5 (NP-completeness):** Show that these fundamental problems are "computationally equivalent" and appear to be different manifestations of one **really hard** problem.

Polynomial-time reduction

Given languages A and B,

$A \leq_p B$

if there is a ***poly-time*** computable function f,

such that for all strings w,

$w \in A$ iff $f(w) \in B$.

Polynomial-time reductions are the major tool we have to understand P and NP

Implication of poly-time reductions

Theorem. If A $\leq_p$ B and B$\in$ **P** then A$\in$ **P**.

(So, if A $\leq_p$ B and A$\notin$ **P** then B$\notin$ **P**.**)**

Theorem. If A $\leq_p$ B and B $\leq_p$ C then A $\leq_p$ C.

(Poly-time reductions compose.)

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Basic reduction strategies

Basic reduction strategies

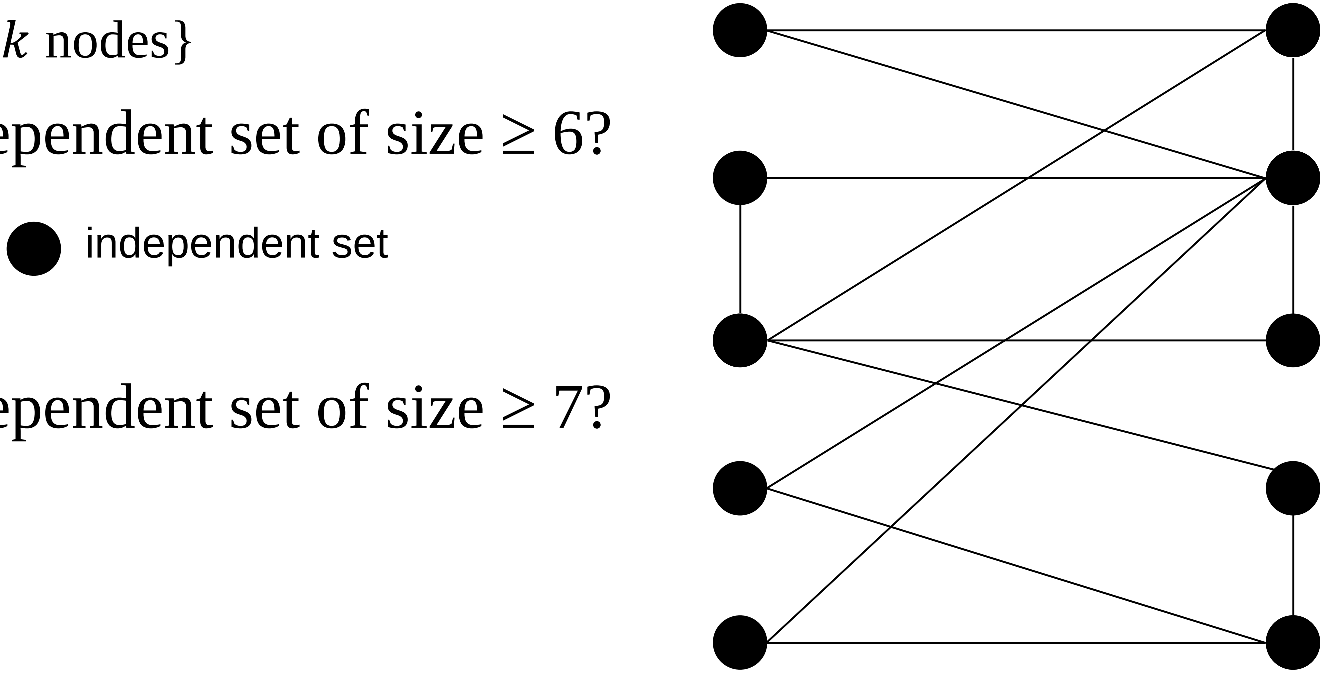
- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Independent Set

Given an undirected graph G, an **independent set** in G is a set of nodes, which includes at most one endpoint of every edge.

INDEPENDENT SET $= \{\langle G, k\rangle \mid G$ is an undirected graph which has an independent set with k nodes$\}$

- Is there an independent set of size ≥ 6?
  – Yes.
- Is there an independent set of size ≥ 7?
  – No.



Vertex Cover

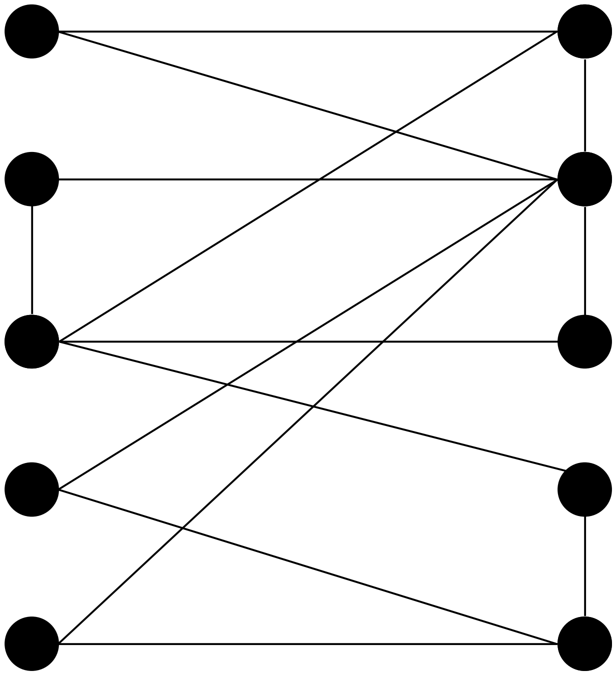
Given an undirected graph G, a **vertex cover** in G is a set of nodes, which includes at ***least*** one endpoint of every edge.

VERTEX COVER $= \{\langle G, k\rangle \mid G$ is an undirected graph which has a vertex cover with k nodes$\}$

- Is there vertex cover of size ≤ 4?
  - Yes.

vertex cover

- Is there a vertex cover of size ≤ 3?
  - No.



Independent Set and Vertex Cover

Claim. S is an independent set iff V – S is a vertex cover.

- $\Rightarrow$
  - Let S be any independent set.
  - Consider an arbitrary edge (u, v).
  - S is independent $\Rightarrow$ $u \notin S$ or $v \notin S$ $\Rightarrow$ $u \in V - S$ or $v \in V - S$.
  - Thus, V – S covers (u, v).
- $\Leftarrow$
  - Let V – S be any vertex cover.
  - Consider two nodes $u \in S$ and $v \in S$.
  - Then $(u, v) \notin E$ since V – S is a vertex cover.
  - Thus, no two nodes in S are joined by an edge $\Rightarrow$ S independent set. ▪

INDEPENDENT SET reduces to VERTEX COVER

Theorem. INDEPENDENT-SET $\leq_p$ VERTEX-COVER.

Proof. "On input $\langle G, k\rangle$, where G is an undirected graph and k is an integer,

1. Output $\langle G, n-k\rangle$, where n is the number of nodes in G."

Correctness:

- G has an independent set of size k iff it has a vertex cover of size $n-k$.
- Reduction runs in linear time.

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Reduction from special case to general case

Basic reduction strategies

- Reduction by simple equivalence.
- Reduction from special case to general case.
- Reduction by encoding with gadgets.

Set Cover

Given a set U, called a *universe*, and a collection of its subsets $S_1, S_2, \ldots, S_m$, a **set cover** of U is a subcollection of subsets whose union is U.

- SET COVER=$\{\langle U, S_1, S_2, \ldots, S_m; k\rangle \mid$

U has a set cover of size $k\}$

U = { 1, 2, 3, 4, 5, 6, 7 }	
k = 2	
S_1 = {3, 7}	S_4 = {2, 4}
S_2 = {3, 4, 5, 6}	S_5 = {5}
S_3 = {1}	S_6 = {1, 2, 6, 7}

- Sample application.
  - m available pieces of software.
  - Set U of n capabilities that we would like our system to have.
  - The ith piece of software provides the set $S_i \subseteq$ U of capabilities.
  - Goal: achieve all n capabilities using fewest pieces of software.

VERTEX COVER reduces to SET COVER

Theorem. VERTEX-COVER $\leq_P$ SET-COVER.

Proof. "On input $\langle G, k \rangle$, where $G = (V, E)$ is an undirected graph and k is an integer,

1. Output $\langle U, S_1, S_2, \ldots, S_m; k \rangle$, where U=E and

$$S_v = \{e \in E \mid e \text{ incident to } v\}"$$

Correctness:

- G has a vertex cover of size k iff U has a set cover of size k.
- Reduction runs in linear time.

Reduction by encoding with gadgets

Basic reduction strategies

- Reduction by simple equivalence.
- Reduction from special case to general case.
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Satisfiability

- **Boolean variables:** variables that can take on values T/F (or 1/0)
- **Boolean operations:** $\vee$, $\wedge$, and $\neg$
- **Boolean formula:** expression with Boolean variables and ops

SAT = $\{\langle\Phi\rangle \mid \Phi$ is a satisfiable Boolean formula$\}$

- **Literal:** A Boolean variable or its negation. x_i or $\overline{x_i}$
- **Clause:** OR of literals. $C_j = x_1 \vee \overline{x_2} \vee x_3$
- **Conjunctive normal form (CNF):** AND of clauses. $\Phi = C_1 \wedge C_2 \wedge C_3 \wedge C_4$

3SAT = $\{\langle\Phi\rangle \mid \Phi$ is a satisfiable Boolean CNF formula, where each clause contains exactly 3 literals$\}$

each corresponds to a different variable

Ex: $(\overline{x_1} \vee x_2 \vee x_3) \wedge (x_1 \vee \overline{x_2} \vee x_3) \wedge (x_2 \vee x_3) \wedge (\overline{x_1} \vee \overline{x_2} \vee \overline{x_3})$

Yes: x_1 = true, x_2 = true x_3 = false.

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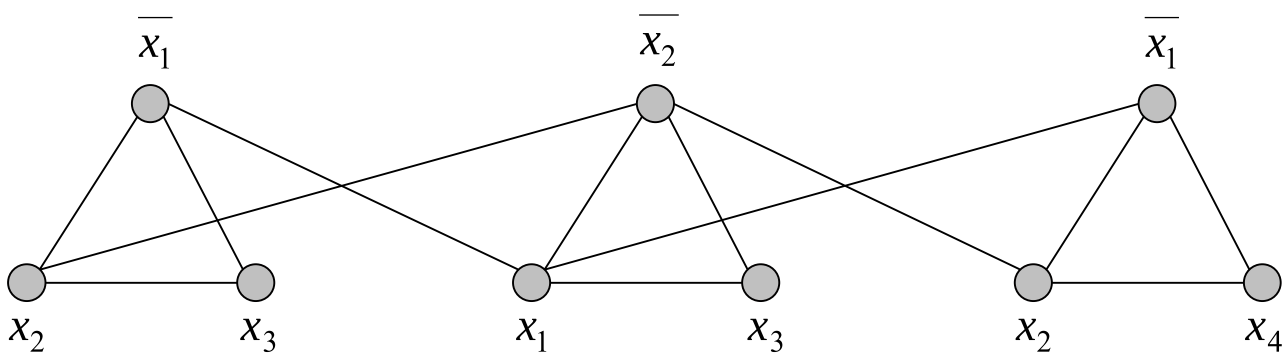
3SAT reduces to INDEPENDENT SET

Theorem. 3-SAT $\leq_P$ INDEPENDENT-SET.

Proof. "On input $\langle \Phi \rangle$, where Φ is a 3CNF formula,

1. Construct graph G from Φ
   - G contains 3 vertices for each clause, one for each literal.
   - Connect 3 literals in a clause in a triangle.
   - Connect literal to each of its negations.
2. Output $\langle G, k \rangle$, where k is the number of clauses in G."

G



k = 3

$$\Phi = \left(\overline{x_1} \vee x_2 \vee x_3\right) \wedge \left(x_1 \vee \overline{x_2} \vee x_3\right) \wedge \left(\overline{x_1} \vee x_2 \vee x_4\right)$$

3SAT reduces to INDEPENDENT SET

Correctness. Let k = # of clauses and ℓ= # of literals in Φ.

Φ is satisfiable iff G contains an independent set of size k.

- $\Rightarrow$ Given satisfying assignment, select one true literal from each triangle. This is an independent set of size k.
- $\Leftarrow$ Let S be an independent set of size k.
  - S must contain exactly one vertex in each triangle.
  - Set these literals to true, and other literals in a consistent way.
  - Truth assignment is consistent and all clauses are satisfied.

Run time. $O(k + \ell^2)$, i.e. polynomial in the input size.

Summary

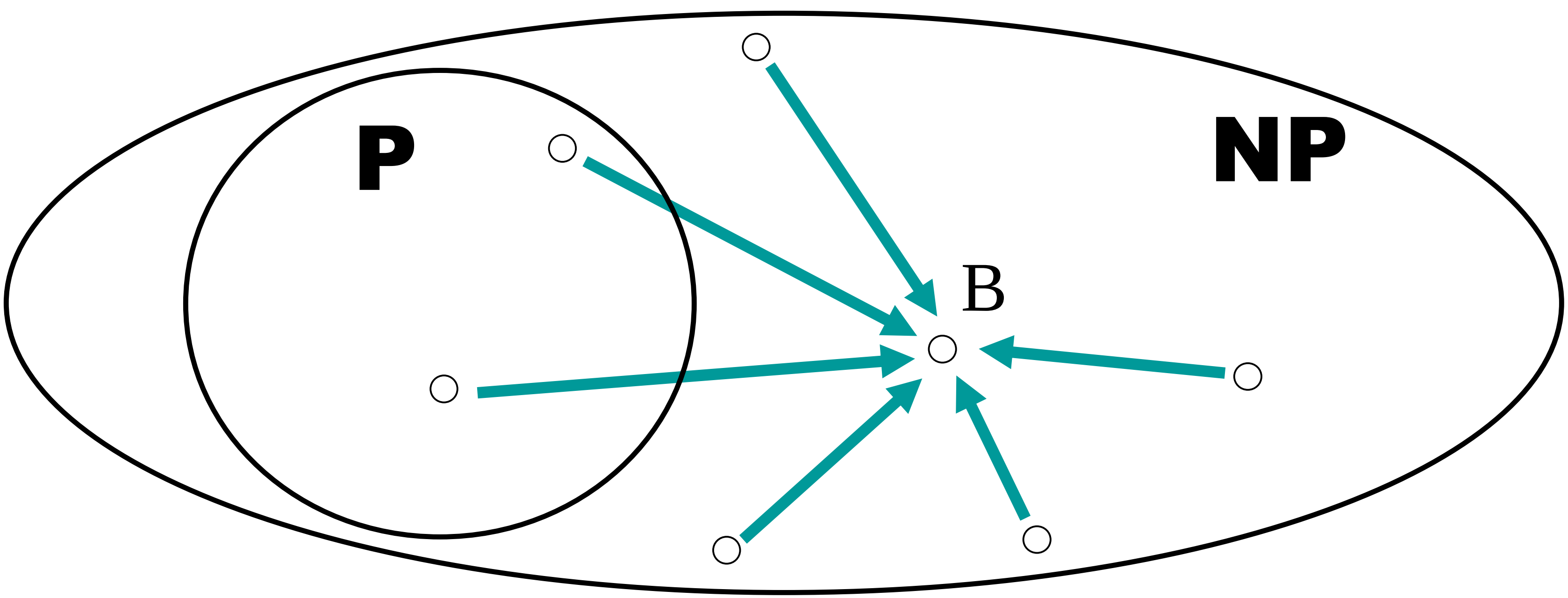
- Basic reduction strategies.
  - Simple equivalence: INDEPENDENT-SET $\equiv_P$ VERTEX-COVER.
  - Special case to general case: VERTEX-COVER $\leq_P$ SET-COVER.
  - Encoding with gadgets: 3-SAT $\leq_P$ INDEPENDENT-SET.

- **Transitivity**. If $X \leq_P Y$ and $Y \leq_P Z$, then $X \leq_P Z$.
- Proof idea. Compose the two algorithms.
- Ex: 3-SAT $\leq_P$ INDEPENDENT-SET $\leq_P$ VERTEX-COVER $\leq_P$ SET-COVER.

Hardest problems in NP

A language B is **NP-complete** if

1. B ∈ **NP**
2. B is **NP-hard**,i.e.,
   every language in NP is poly-time reducible to B.

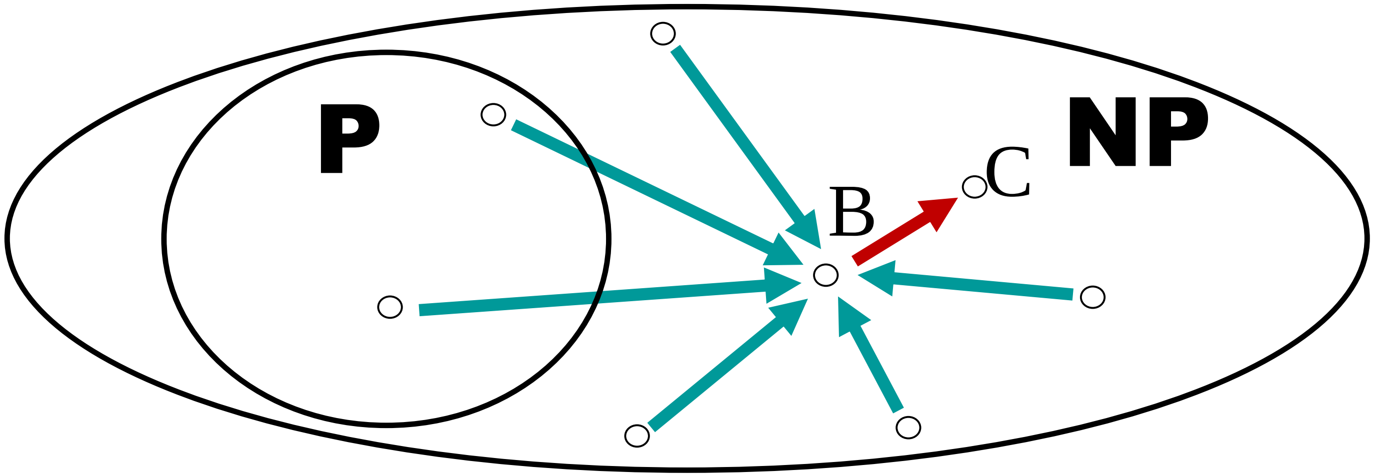


Implication of poly-time reductions

Theorem. If

- B is **NP**-complete,
- C$\in$ **NP** and
- B$\leq_p$C

then C is **NP**-complete.



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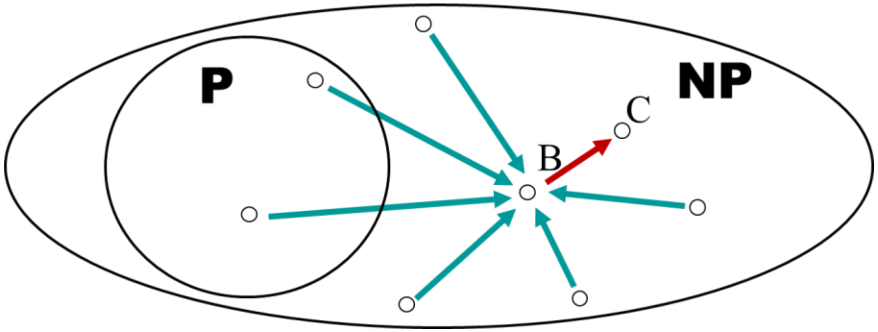
Implication of poly-time reductions

Theorem. If
- B is **NP**-complete,
- C∈ **NP** and
- $B \leq_p C$

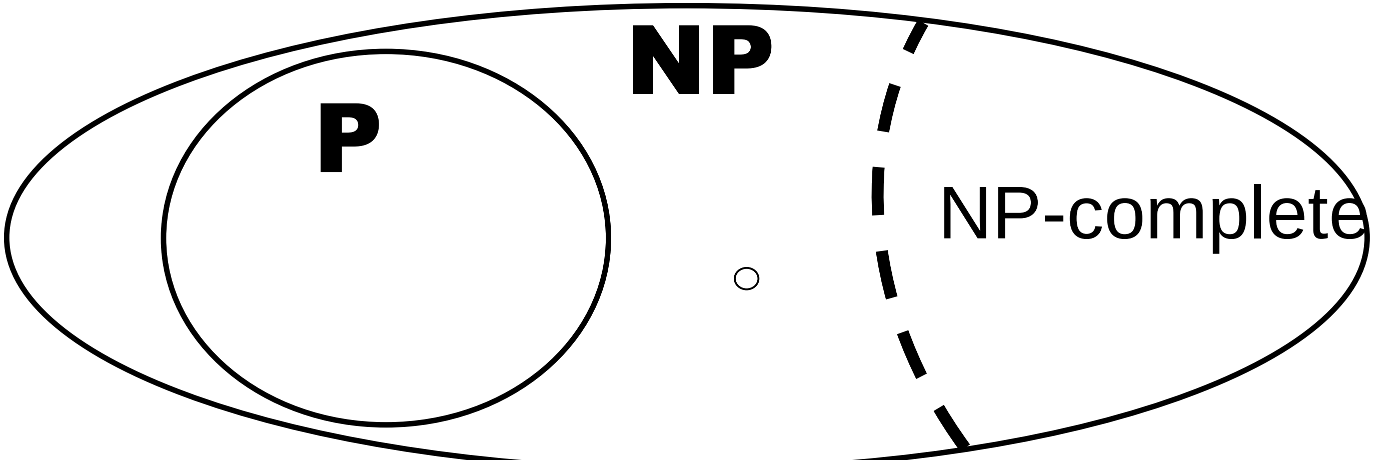
then C is **NP**-complete.



Theorem. If B is **NP**-complete and B∈ **P** then

$$\mathbf{P} = \mathbf{NP}.$$

(So, if B is **NP**-complete and $\mathbf{P} \neq \mathbf{NP}$
then there is no poly-time algorithm for B.)


NP
P
NP-complete