Intro to Theory of Computation

CS 332

LECTURE 26

Last time

- Space complexity
- The class PSPACE

Today

- PSPACE-completeness
- Hierarchy theorems

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Review question

Let $f(n)$ be a function, where $f(n) \geq n$.

If a language A is decided by a NTM that runs in space $f(n)$ then A is decided by a TM that runs in space O($f^2(n)$).

A. True

B. False

C. Unknown

D. None of the above

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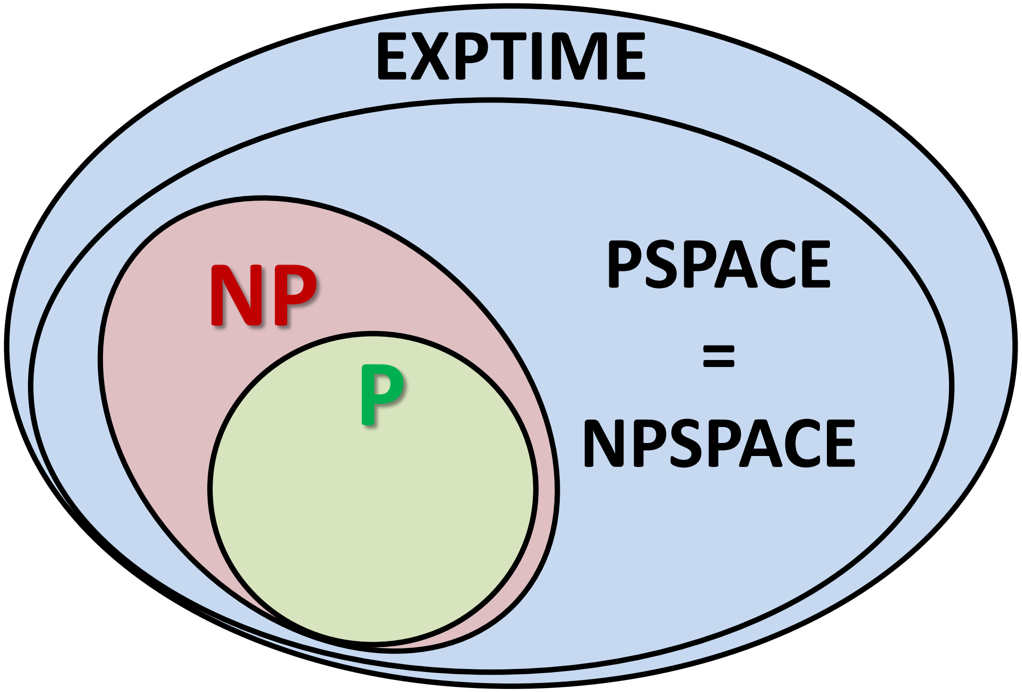
Review question

Relationships between the following classes are an open question

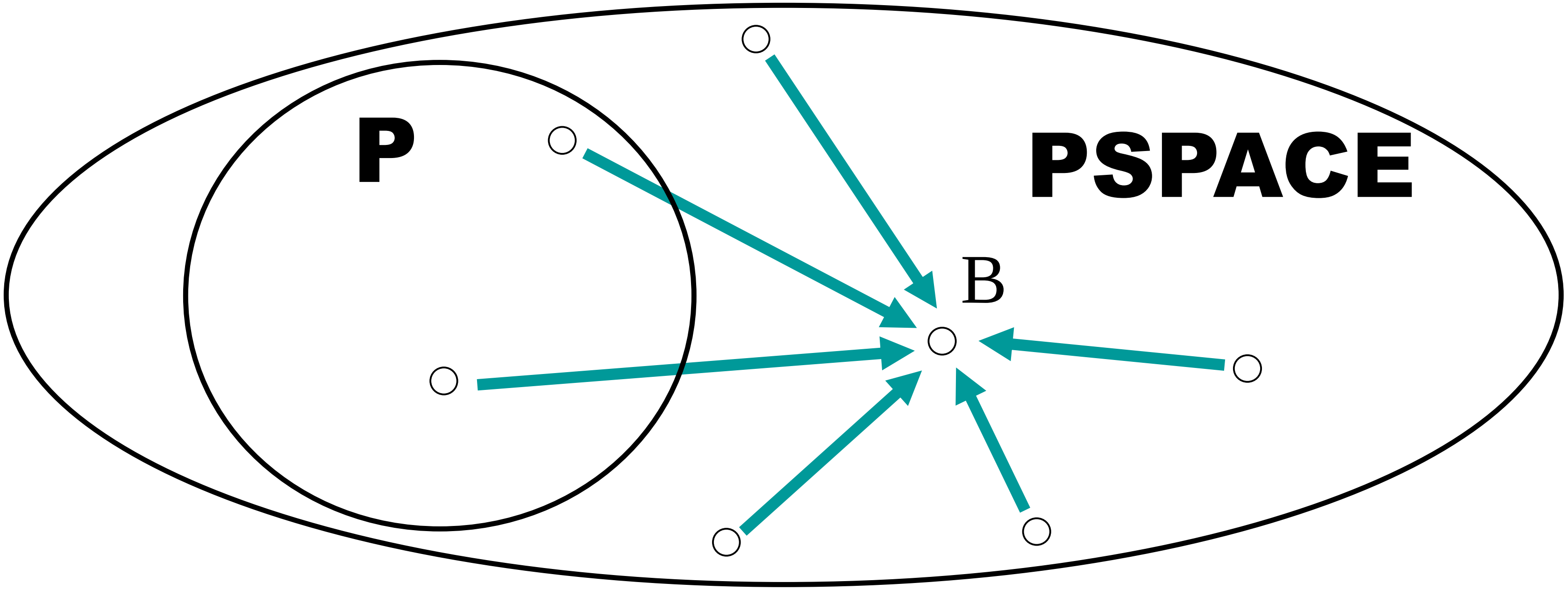


A. P vs NP
B. P vs PSPACE
C. NP vs EXPTIME
D. All of the above
E. All of the above and P vs EXPTIME

Hardest problems in PSPACE

A language B is **PSPACE-complete** if

1. B∈ **PSPACE**
2. B is **PSPACE-hard**,i.e.,
   every language in PSPACE is poly-time reducible to B.



The TQBF problem

- **Boolean variables:** variables that can take on values T/F (or 1/0)
- **Boolean operations:** $\vee$, $\wedge$, and $\neg$
- **Boolean formula:** expression with Boolean variables and ops
- **Quantified Boolean formula:** Boolean formula with quantifiers ($\forall, \exists$)
- **Fully Quantified Boolean formula:** all variables have quantifiers ($\forall, \exists$)

We only consider the form where all quantifiers appear in the beginning.

Ex. $\forall x \exists y [(x \vee y) \wedge (\bar{x} \vee \bar{y})]$ **True**

$\exists y \forall x \, [(x \vee y) \wedge (\bar{x} \vee \bar{y})]$ **False**

- Each fully quantified Boolean formula is either true or false.
- The order of quantifiers matters!

TQBF = $\{\langle \phi \rangle \mid \phi$ is a true fully quantified Boolean formula$\}$

TQBF is PSPACE-complete

1. **TQBF is in PSPACE**
2. **TQBF is PSPACE-hard**

Prove: TQBF ∈ PSPACE

T = `` On input $\langle\phi\rangle$,
where ϕ is a fully quantified Boolean formula:

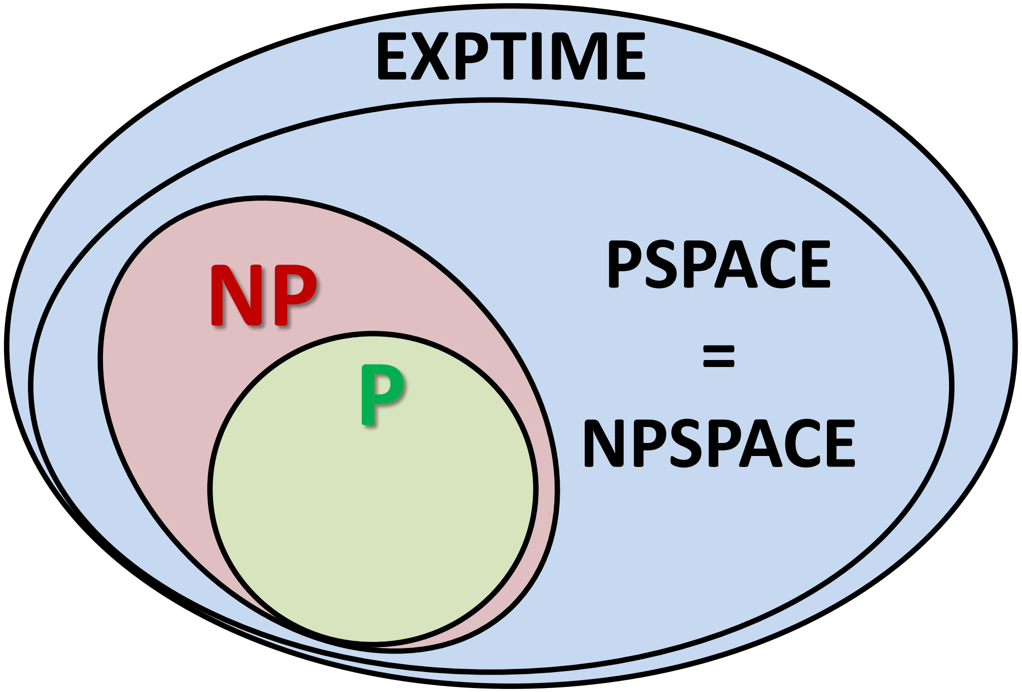
1. If ϕ has no quantifiers, it has only constants (and no variables). Evaluate ϕ.
   If true, accept; o.w., reject.
2. If ϕ is of the form $\exists x\,\psi$, recursively call T on ψ with $x = 0$ and then on ψ with $x = 1$.
   If either call accepts, accept; o.w., reject.
3. If ϕ is of the form $\forall x\,\psi$, recursively call T on ψ with $x = 0$ and then on ψ with $x = 1$.
   If both calls accept, accept; o.w., reject.”

- If n is the input length, T uses space $O(n)$.

Question

If TQBF is in P then it implies that

A. P = NP
B. P = PSPACE
C. P = EXPTIME
D. (A) and (B) are true
E. (A), (B), (C) are true



TQBF as a game

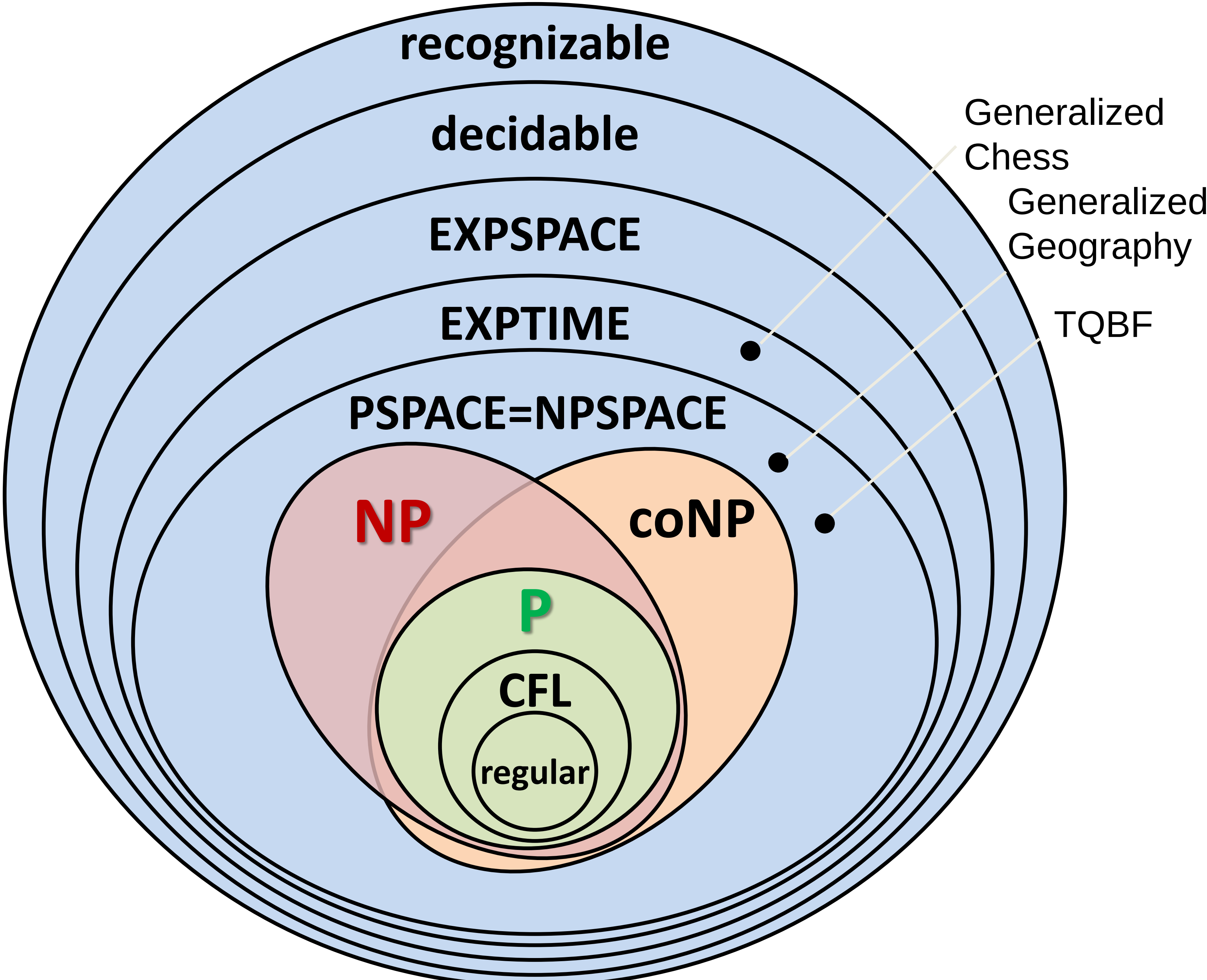
- Two players: E (for ∃) and A (for ∀)
- Given: a quantified Boolean formula
- Players alternate choosing values for variables (in the same order as the quantifiers appear).
- E wins if the formula evaluates to 1.

- Does E have a **winning strategy** in the game? (Can she win if both E and A play optimally?)

Winning strategies for games

Determining whether there is a winning strategy for a specified game position is hard for many games

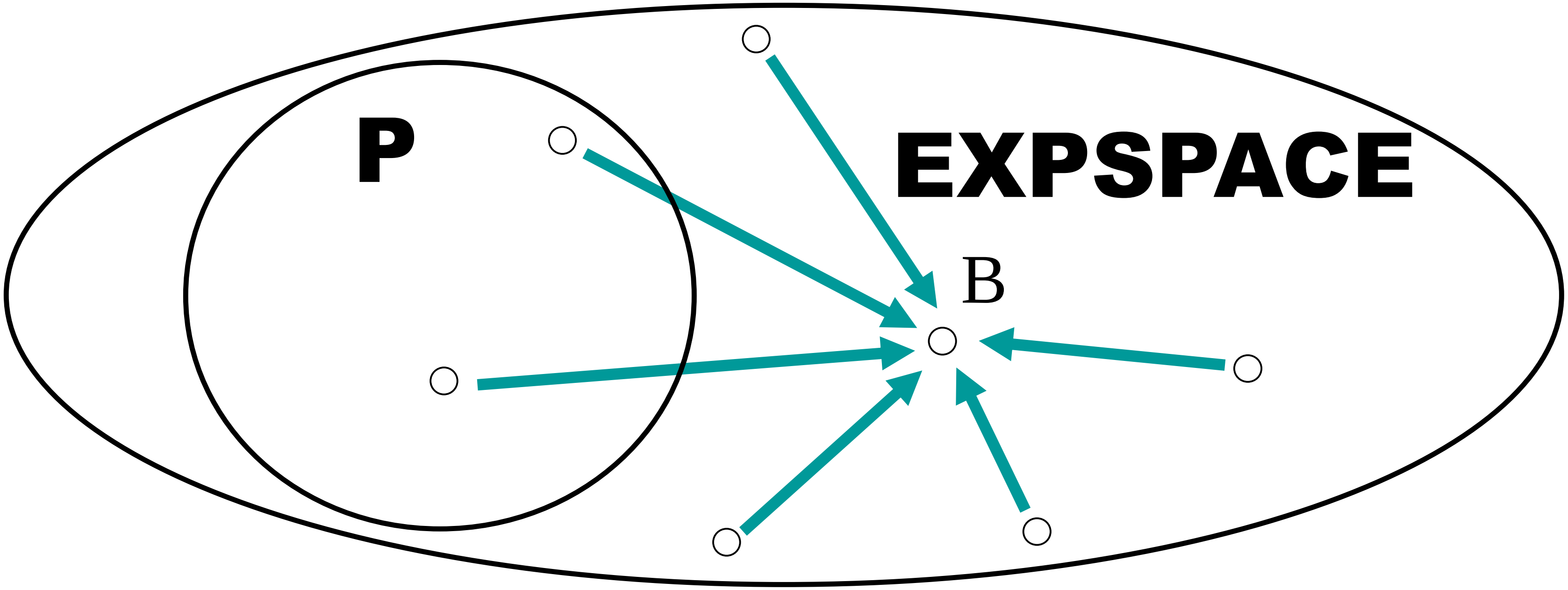
- Generalized Geography is PSPACE-complete
- Generalized Chess is EXPTIME-complete


recognizable
decidable
EXPSPACE
EXPTIME
PSPACE=NPSPACE
NP
coNP
P
CFL
regular
Generalized Chess
Generalized Geography
TQBF

Hardest problems in EXPSPACE

A language B is **EXPSPACE-complete** if

1. B∈ **EXPSPACE**
2. B is **EXPSPACE-hard**,i.e., every language in EXPSPACE is poly-time reducible to B.

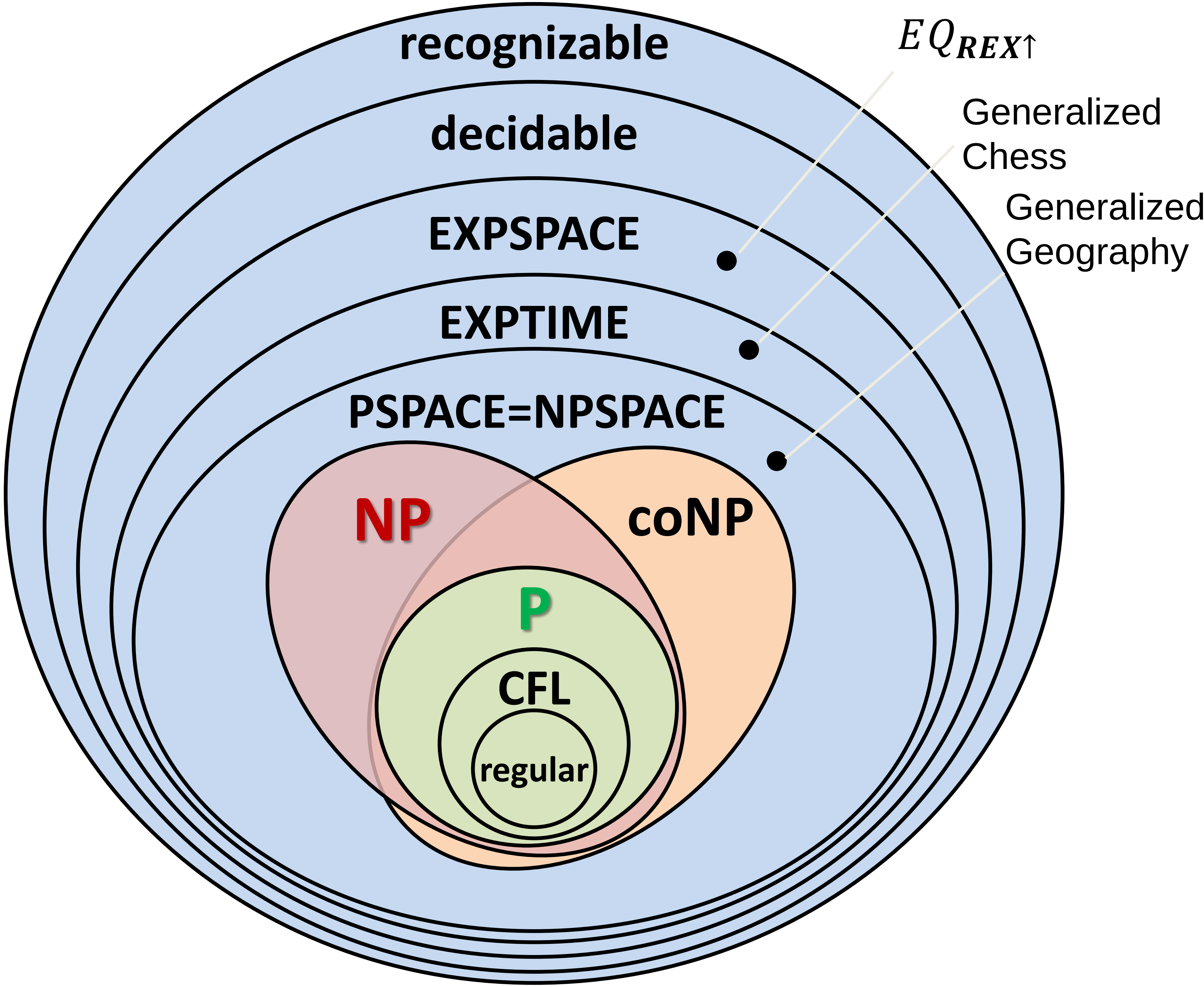


The $EQ_{\boldsymbol{REX}\uparrow}$ problem

- **Regular operations:** $\cup, \ \circ, \ *$
- **Regular expressions: use regular operations, $\emptyset, \boldsymbol{\varepsilon}, \boldsymbol{\Sigma}$, members of $\boldsymbol{\Sigma}$**
- **We can test equivalence of two regular expressions in poly space**
- **Exponentiation operation:** $R^k = R \uparrow k = R \circ \cdots \circ R$ (k times)

$EQ_{\boldsymbol{REX}\uparrow}$ **= {⟨$\boldsymbol{Q}, \boldsymbol{R}$⟩ | $\boldsymbol{Q}$ and R are equivalent regular expressions with exponentiation}**

$EQ_{\boldsymbol{REX}\uparrow}$ **is EXPSPACE-complete.**



$EQ_{REX\uparrow}$
Generalized Chess
Generalized Geography
recognizable
decidable
EXPSPACE
EXPTIME
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regular

Motivating question

Is there a decidable language which is provably not in P?

Hierarchy Theorems

A function $f\colon \mathbb{N} \to \mathbb{N}$ is **space constructible** if the function that maps the string 1^n to the binary representation of $f(n)$ is computable in space $O\big(f(n)\big)$.

- Fractional functions are rounded down.
- Examples: $\log_2 n$, $\sqrt{n}$, $n \log_2 n$, n^2, 2^n,

all commonly occurring functions that are $\Omega(\log n)$

Hierarchy Theorems

Space Hierarchy Theorem

For any space constructible function f,

a language exists that is decidable in $O\big(f(n)\big)$ space,

but not in $o(f(n))$ space.

- **Corollary 1:** For all $\alpha, \beta \geq 0$, where $\alpha < \beta$,

  SPACE(n^{α}) is a strict subset of SPACE(n^{β}).
- **Corollary 2:** PSPACE is a strict subset of EXPSPACE, where

  EXPSPACE=$\cup_k SPACE(2^{n^k})$
- **Corollary 3:** $EQ_{\boldsymbol{REX\uparrow}}$ is not in PSPACE.

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Hierarchy Theorems

A function $f\colon \mathbb{N} \to \mathbb{N}$ is **time constructible** if the function that maps the string 1^n to the binary representation of $f(n)$ is computable in time $O(f(n))$.

- Fractional functions are rounded down.
- Examples: $n \log_2 n$, n^2, 2^n,

all commonly occurring functions that are $\Omega(n \log n)$

Hierarchy Theorems

Time Hierarchy Theorem

For any time constructible function f,

a language exists that is decidable in $O\big(f(n)\big)$ time,

but not in $o\left(\frac{f(n)}{\log f(n)}\right)$ time.

- **Corollary 1:** For all $\alpha, \beta \geq 0$, where $\alpha < \beta$,
  TIME(n^{α}) is a strict subset of TIME(n^{β}).
- **Corollary 2:** P is a strict subset of EXPTIME
- **Corollary 3:** Generalized Chess is not in P.

Review question

3SAT is decidable, but not in polynomial time.

A. True
B. False
C. Unknown
D. None of the above

Review question

$EQ_{REX\uparrow}$ **is decidable, but not in polynomial time.**

A. **True**
B. **False**
C. **Unknown**
D. **None of the above**

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Review question

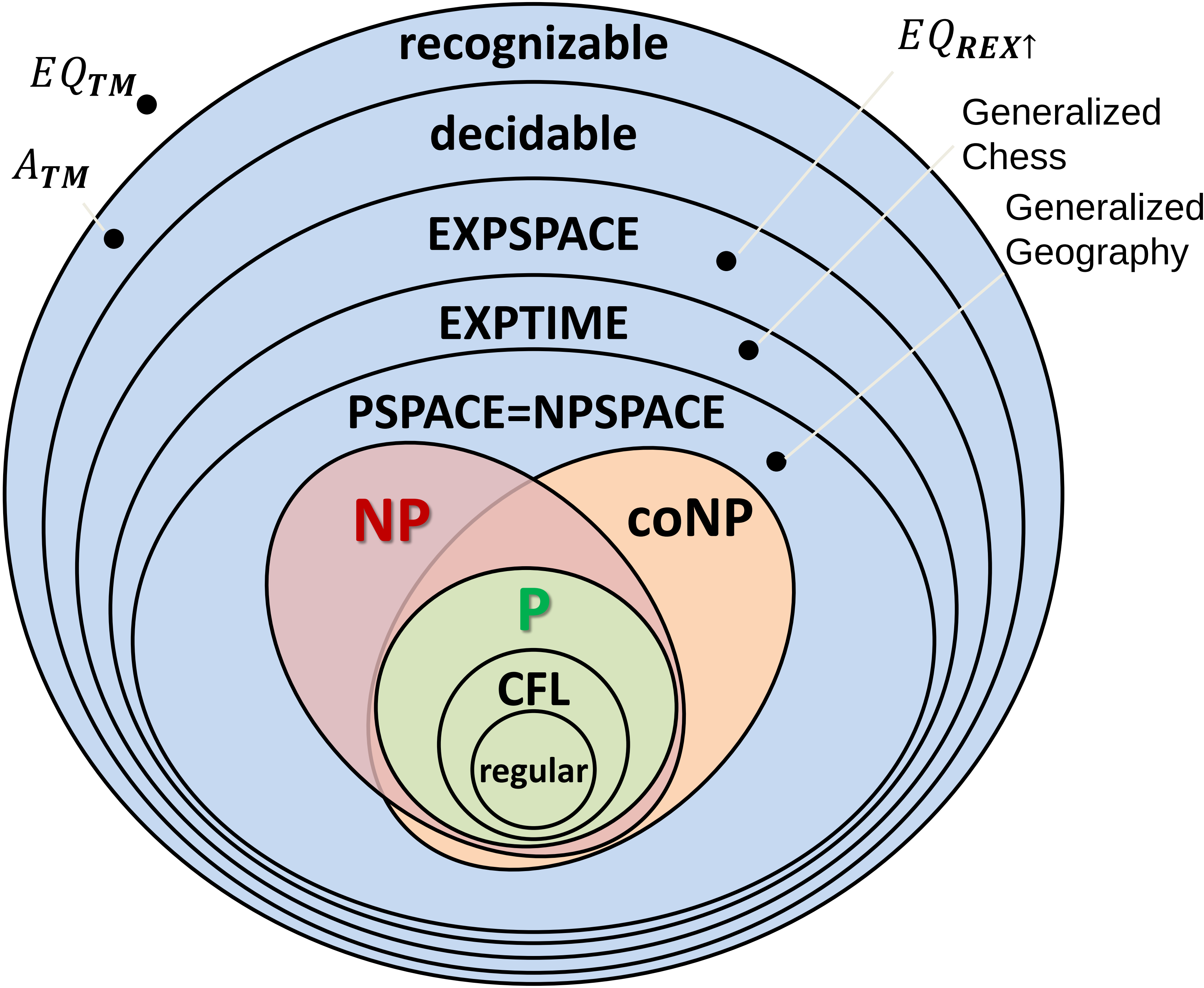
A_{TM} is decidable, but not in polynomial time.

A. True

B. False

C. Unknown

D. None of the above



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EQ_{TM}
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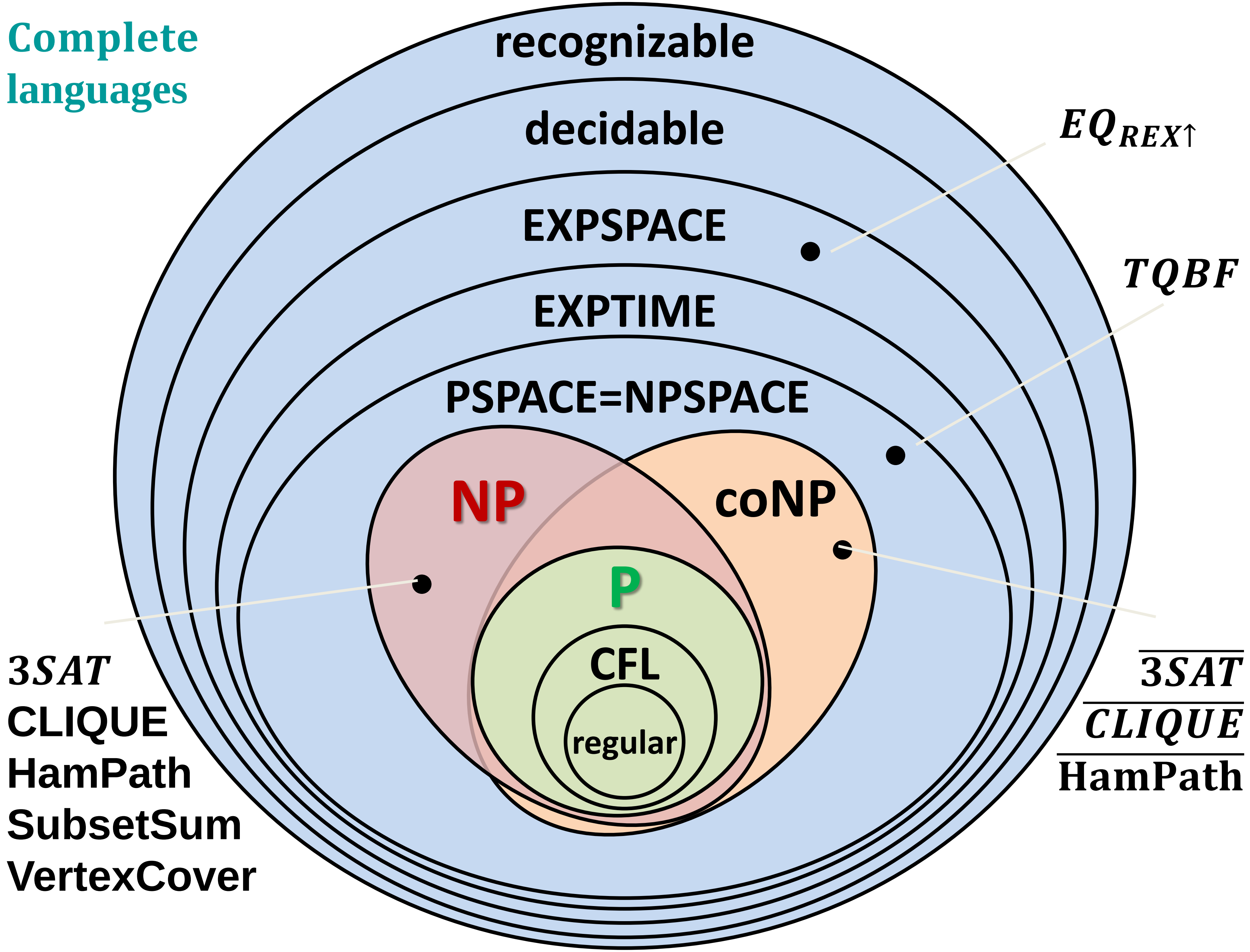
Generalized Chess
Generalized Geography


Complete languages
recognizable
decidable
EXPSPACE
EXPTIME
PSPACE=NPSPACE
NP
coNP
P
CFL
regular
$EQ_{REX\uparrow}$
$TQBF$
$3SAT$
CLIQUE
HamPath
SubsetSum
VertexCover
$\overline{3SAT}$
$\overline{CLIQUE}$
$\overline{\text{HamPath}}$

Course parts

1. Automata theory
2. Computability theory
3. Complexity theory