INTRO TO THEORY OF COMPUTATION

LECTURE 26

Last time
- Space complexity
- The class PSPACE

Today
- PSPACE-completeness
- Hierarchy theorems
Let $f(n)$ be a function, where $f(n) \geq n$.

If a language $A$ is decided by a NTM that runs in space $f(n)$ then $A$ is decided by a TM that runs in space $O(f^2(n))$.

A. True
B. False
C. Unknown
D. None of the above
Review question

Relationships between the following classes are an open question

A. P vs NP
B. P vs PSPACE
C. NP vs EXPTIME
D. All of the above
E. All of the above and P vs EXPTIME
A language B is **PSPACE-complete** if

1. $B \in \text{PSPACE}$
2. B is **PSPACE-hard**, i.e.,
   every language in PSPACE is poly-time reducible to B.
The TQBF problem

- **Boolean variables**: variables that can take on values T/F (or 1/0)
- **Boolean operations**: \( \lor, \land, \text{ and } \lnot \)
- **Boolean formula**: expression with Boolean variables and ops
- **Quantified Boolean formula**: Boolean formula with quantifiers (\( \forall, \exists \))
- **Fully Quantified Boolean formula**: all variables have quantifiers (\( \forall, \exists \))
  
  We only consider the form where all quantifiers appear in the beginning.

\[
\text{Ex. } \forall x \exists y [(x \lor y) \land (\overline{x} \lor \overline{y})] \quad \text{True} \\
\exists y \forall x [(x \lor y) \land (\overline{x} \lor \overline{y})] \quad \text{False}
\]

- Each fully quantified Boolean formula is either true or false.
- The order of quantifiers matters!

\[
\text{TQBF} = \{ \langle \phi \rangle \mid \phi \text{ is a true fully quantified Boolean formula} \} 
\]
TQBF is PSPACE-complete

1. TQBF is in PSPACE
2. TQBF is PSPACE-hard
Prove: $T \in \text{PSPACE}$

$T = \text{`` On input } \langle \phi \rangle, \text{ where } \phi \text{ is a fully quantified Boolean formula:} $

1. If $\phi$ has no quantifiers, it has only constants (and no variables). Evaluate $\phi$.
   If true, accept; o.w., reject.
2. If $\phi$ is of the form $\exists x \psi$, recursively call $T$ on $\psi$ with $x = 0$ and then on $\psi$ with $x = 1$.
   If either call accepts, accept; o.w., reject.
3. If $\phi$ is of the form $\forall x \psi$, recursively call $T$ on $\psi$ with $x = 0$ and then on $\psi$ with $x = 1$.
   If both calls accept, accept; o.w., reject.”

• If $n$ is the input length, $T$ uses space $O(n)$. 
Question

If TQBF is in P then it implies that

A. $P = NP$
B. $P = \text{PSPACE}$
C. $P = \text{EXPTIME}$
D. (A) and (B) are true
E. (A), (B), (C) are true
TQBF as a game

- Two players: E (for ∃) and A (for ∀)
- Given: a quantified Boolean formula
- Players alternate choosing values for variables (in the same order as the quantifiers appear).
- E wins if the formula evaluates to 1.

• Does E have a winning strategy in the game? (Can she win if both E and A play optimally?)
Determining whether there is a winning strategy for a specified game position is hard for many games

- Generalized Geography is PSPACE-complete
- Generalized Chess is EXPTIME-complete
recognizable

decidable

EXPSPACE

EXPTIME

PSPACE=NPSPACE

NP

coNP

P

CFL

regular

Generalized Chess

Generalized Geography

TQBF
A language $B$ is \textbf{EXPSPACE-complete} if

1. $B \in \text{EXPSPACE}$
2. $B$ is \textbf{EXPSPACE-hard}, i.e., every language in \text{EXPSPACE} is poly-time reducible to $B$. 
The $EQ_{REX}$ problem

- **Regular operations:** $\cup$, $\circ$, $\ast$
- **Regular expressions:** use regular operations, $\emptyset$, $\varepsilon$, $\Sigma$, members of $\Sigma$
- We can test equivalence of two regular expressions in poly space
- **Exponentiation operation:** $R^k = R \uparrow k = R \circ \cdots \circ R$ ($k$ times)

$EQ_{REX} = \{ \langle Q, R \rangle \mid Q \text{ and } R \text{ are equivalent regular expressions with exponentiation} \}$

$EQ_{REX}$ is EXPSPACE-complete.
recognizable

decidable

EXPSPACE

EXPTIME

PSPACE=NPSPACE

P

CFL

regular

NP

coNP

Generalized Chess

Generalized Geography

$E Q_{REX}^+$
Motivating question

Is there a decidable language which is provably not in P?
A function $f: \mathbb{N} \rightarrow \mathbb{N}$ is **space constructible** if the function that maps the string $1^n$ to the binary representation of $f(n)$ is computable in space $O(f(n))$.

- Fractional functions are rounded down.
- **Examples:** $\log_2 n, \sqrt{n}, n \log_2 n, n^2, 2^n$, all commonly occurring functions that are $\Omega(\log n)$
Hierarchy Theorems

Space Hierarchy Theorem
For any space constructible function $f$, a language exists that is decidable in $O(f(n))$ space, but not in $o(f(n))$ space.

- **Corollary 1:** For all $\alpha, \beta \geq 0$, where $\alpha < \beta$, \(\text{SPACE}(n^\alpha)\) is a strict subset of \(\text{SPACE}(n^\beta)\).

- **Corollary 2:** PSPACE is a strict subset of EXPSPACE, where \(\text{EXPSPACE} = \bigcup_k \text{SPACE}(2^{n^k})\).

- **Corollary 3:** $E Q_{REX}^\uparrow$ is not in PSPACE.
A function $f : \mathbb{N} \to \mathbb{N}$ is **time constructible** if the function that maps the string $1^n$ to the binary representation of $f(n)$ is computable in time $O(f(n))$.

- Fractional functions are rounded down.
- **Examples:** $n \log_2 n$, $n^2$, $2^n$,

all commonly occurring functions that are $\Omega(n \log n)$
Hierarchy Theorems

Time Hierarchy Theorem

For any time constructible function $f$, a language exists that is decidable in $O(f(n))$ time, but not in $o\left(\frac{f(n)}{\log f(n)}\right)$ time.

- **Corollary 1**: For all $\alpha, \beta \geq 0$, where $\alpha < \beta$,
  $\text{TIME}(n^\alpha)$ is a strict subset of $\text{TIME}(n^\beta)$.
- **Corollary 2**: P is a strict subset of EXP\-TIME
- **Corollary 3**: Generalized Chess is not in P.
3SAT is decidable, but not in polynomial time.

A. True
B. False
C. Unknown
D. None of the above
Review question

\[ EQ_{REX}^{\uparrow} \text{ is decidable, but not in polynomial time.} \]

A. True
B. False
C. Unknown
D. None of the above
Review question

\( A_{\text{TM}} \) is decidable, but not in polynomial time.

A. True
B. False
C. Unknown
D. None of the above
recognizable

decidable

EXPSPACE

EXPTIME

PSPACE=NPSPACE

P

coNP

CFL

regular

Generalized Chess

Generalized Geography
recognizable

decidable

EXPSPACE

EXPTIME

PSPACE=NPSPACE

P

NP

coNP

CFL

regular

3SAT

CLIQUE

HamPath

SubsetSum

VertexCover

Complete languages

$\text{EQ}_{\text{REX}}$↑

$TQBF$

3SAT

CLIQUE

HamPath

3SAT

CLIQUE

HamPath
Course parts

1. Automata theory
2. Computability theory
3. Complexity theory