Recall: random variables

- A random variable X on a sample space Ω is a function $X: \Omega \to \mathbb{R}$ that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.
- For each random variable, we should understand:
  - The set of values it can take.
  - The probabilities with which it takes on these values.
- The distribution of a discrete random variable X is the collection of pairs $\{(a, \Pr[X = a])\}$.

Question

You roll two dice. Let X be the random variable that represents the sum of the numbers you roll.

What is the probability of the event X=6?

A. 1/36

B. 1/9

C. 5/36

D. 1/6

E. None of the above.

Question

You roll two dice. Let X be the random variable that represents the sum of the numbers you roll.

How many different values can X take on?

A. 6

B. 11

C. 12

D. 36

E. None of the above.

Question

You roll two dice. Let X be the random variable that represents the sum of the numbers you roll.

What is the distribution of X?

A. Uniform distribution on the set of possible values.

B. It satisfies $\Pr[X=2] < \Pr[X=3] < \cdots < \Pr[X=12]$.

C. It satisfies $\Pr[X=2] > \Pr[X=3] > \cdots > \Pr[X=12]$.

D. It satisfies $\Pr[X=2] < \Pr[X=3] < \cdots < \Pr[X=7]$ and $\Pr[X=7] > \Pr[X=8] > \cdots > \Pr[X=12]$.

E. None of the above is true.

Independent RVs: definition

- Random variables X and Y are independent if
$$\Pr[(X = x) \cap (Y = y)] = \Pr[X = x] \cdot \Pr[Y = y]$$
for all values x and y.

- Random variables $X_1, X_2, \ldots, X_n$ are mutually independent if for all subsets of $I \subseteq [n]$ and all values x_i, where $i \in I$,
$$\Pr[\cap_{i \in I} (X_i = x_i)] = \prod_{i \in I} \Pr[X_i = x_i].$$

Question

You roll one die. Let X be the random variable that represents the result.

What value does X take, on average?

A. 1/6

B. 3

C. 3.5

D. 6

E. None of the above.

Random variables: expectation

- The expectation of a discrete random variable X over a sample space Ω is

$$E[X] = \sum_{\omega\in\Omega} X(\omega) \cdot \Pr[\omega].$$

- We can group together outcomes ω for which $X(\omega) = a$:

$$E[X] = \sum_{a} a \cdot \Pr[X = a],$$

where the sum is over all possible values a taken by X.

- The second equality is more useful for calculations.

Example: random hats

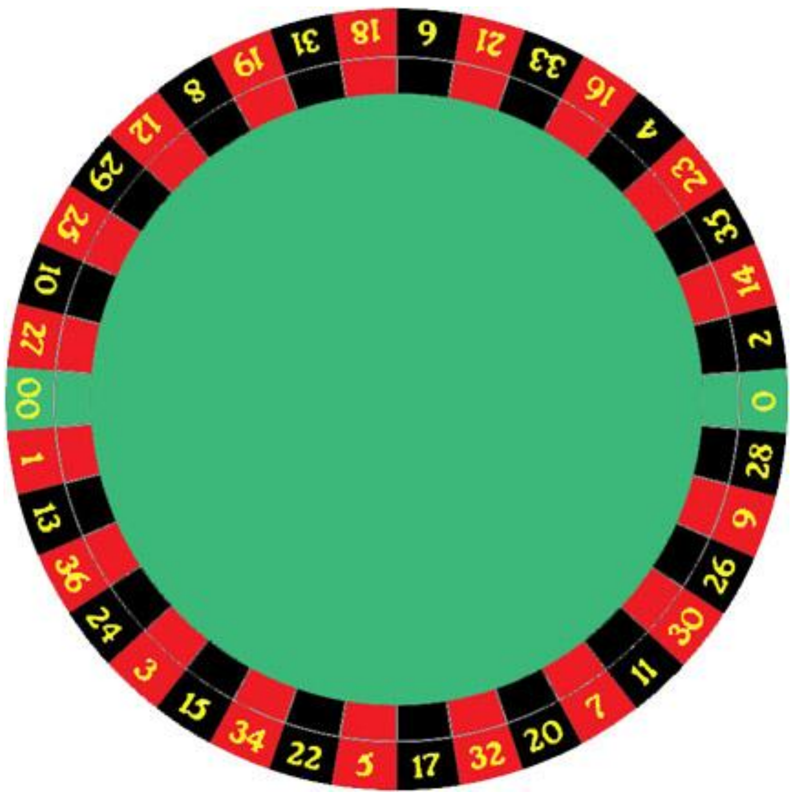
- Example: permutations
  - n students exchange their hats, so that everybody gets a random hat
  - R.V. X: the number of students that got their own hats.
  - E.g., if students 1,2,3 got hats 2,1,3 then X=1.
- Distribution of X:

$$\Pr[X = 0] = \frac{1}{3}, \Pr[X = 1] = \frac{1}{2}, \Pr[X = 3] = \frac{1}{6}.$$

- What's the expectation of X?

Example: roulette

- 38 slots: 18 black, 18 red, 2 green.



- If we bet $1 on red, we get $2 back if red comes up. What’s the expected value of our winnings?

Linearity of expectation

- Theorem. For any two random variables X and Y on the same probability space,

$$E[X+Y]=E[X]+E[Y].$$

Also, for all $c\in\mathbb{R}$,

$$E[cX]=c\cdot E[X].$$

Indicator random variables

- An indicator random variable takes on two values: 0 and 1.
- Lemma. For an indicator random variable X,
$$E[X] = \Pr[X = 1].$$

Question

You have a coin with bias 3/4 (the bias is the probability of HEADS). Let X be the number of HEADS in 1000 tosses of your coin.

You represent X as the sum: $X = X_1 + X_2 + \cdots + X_{1000}$.

What is X_1?

A. 3/4.

B. The number of HEADS.

C. The probability of HEADS in toss 1.

D. The number of heads in toss 1.

E. None of the above.

Question

You have a coin with bias 3/4 (the bias is the probability of HEADS). Let X be the number of HEADS in 1000 tosses of your coin.

What is the expectation of X?

A. 3/4.

B. 4/3.

C. 500.

D. 750.

E. None of the above.