LAST TIME
- Course information
- Verifying polynomial identities (on the blackboard)
- Probability amplification (on the blackboard)
- Probability review

TODAY
- Conditional probability
- Total probability law
- Bayes’ law
### Conditional Probability

The conditional probability of event \( E \) given event \( F \) is

\[
\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)}.
\]

- When \( E \) and \( F \) are independent,

\[
\Pr(E \mid F) = \frac{\Pr(E \cap F)}{\Pr(F)} = \frac{\Pr(E) \cdot \Pr(F)}{\Pr(F)} = \Pr(E).
\]
Review question

- Event A: the numbers on two dice sum to 8.
- Event B: the numbers on two dice are both even.
- What is the probability of A given that B occurred, \( \Pr[A|B] \)?

A. Less than 1/3
B. 1/3
C. Greater than 1/3, but less than 2/3
D. 2/3
E. Greater than 2/3
Card dealing

We deal two cards. What is the probability that the second card is an ace, given that the first is an ace?

A. 3/52
B. 3/51
C. 4/52
D. 5/52
E. None of the answers above are correct.
Independence: definition

• Events A and B are independent if
  \[ \Pr[A \cap B] = \Pr[A] \cdot \Pr[B]. \]

• Equivalent statements (for non-zero probability events A, B):
  \[ \Pr[A|B] = \Pr[A]. \]
  \[ \Pr[B|A] = \Pr[B]. \]
Mutual and pairwise independence

- Events $A_1, \ldots, A_n$ are **mutually independent** if for every subset $I \subseteq \{1, \ldots, n\}$,

$$\Pr \left[ \bigcap_{i \in I} A_i \right] = \prod_{i \in I} \Pr[A_i].$$

- **Implication:** For mutually independent events $A_1, \ldots, A_n$, for all $i \in \{1, \ldots, n\}$ and all subsets $I \subseteq \{1, \ldots, n\} - \{i\}$

$$\Pr[A_i \mid \bigcap_{j \in I} A_j] = \Pr[A_i].$$

- If all pairs of events among $A_1, \ldots, A_n$ are independent then $A_1, \ldots, A_n$ are **pairwise independent**.
Mutual and pairwise independence

• Events $A_1, \ldots, A_n$ are mutually independent if for every subset $I \subseteq \{1, \ldots, n\}$,

$$\Pr\left[\bigcap_{i \in I} A_i\right] = \prod_{i \in I} \Pr[A_i].$$

• If all pairs of events among $A_1, \ldots, A_n$ are independent then $A_1, \ldots, A_n$ are pairwise independent.

• Pairwise independence does not necessarily imply mutual independence!
Independence: example

• We toss a fair coin twice.
• Let A be the event that the 1\textsuperscript{st} flip is HEADS.
• Let B be the event that the 2\textsuperscript{nd} flip is HEADS.
• Let C be the event that both flips are the same.

Are events A, B, C pairwise independent?
Are they mutually independent?
Product rule

• For all events $A, B$,
  \[ \Pr[A \cap B] = \Pr[A] \cdot \Pr[B|A]. \]

• More generally, for all events $A_1, \ldots, A_n$,
  \[ \Pr[\bigcap_{i=1}^{n} A_i] = \Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \ldots \cdot \Pr[A_n|\bigcap_{i=1}^{n-1} A_i] \]
Toss a fair coin three times.
Let $A_i$ be the event that the $i$-th toss is HEADS.
Let $A = A_1 \cap A_2 \cap A_3$.

What is the probability of $A$?

A. $\Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2]$
B. $\Pr[A_1] \cdot \Pr[A_2] \cdot \Pr[A_3]$
C. Both A and B are correct.
D. Neither A nor B is correct.
Review question

Toss a coin that is biased with heads probability $p$ three times.

Let $A_i$ be the event that the $i$-th toss is HEADS.

Let $A = A_1 \cap A_2 \cap A_3$.

What is the probability of $A$?

A. $\Pr[A_1] \cdot \Pr[A_2|A_1] \cdot \Pr[A_3|A_1 \cap A_2]$

B. $\Pr[A_1] \cdot \Pr[A_2] \cdot \Pr[A_3]$

C. Both A and B are correct.

D. Neither A nor B is correct.
Useful conditional probability facts

• **Bayes’ law.** Let A and B be events, and suppose that \( \Pr[B] \neq 0 \). Then

\[
\Pr[A|B] = \frac{\Pr[B|A] \cdot \Pr[A]}{\Pr[B]}.
\]

• **Total probability law.** Let A and B be events. Then

\[
\Pr[B] = \Pr[B|A] \cdot \Pr[A] + \Pr[B|\overline{A}] \cdot \Pr[\overline{A}].
\]
Review question: medical test

5% of people are affected by disorder. The test has 10% false negative rate and 20% false positive rate. What’s the probability that the person who tested positive has the disorder?

We choose a random person from the population.

- Event $A$: the chosen person is affected.
- Event $B$: the chosen person tests positive.

$\Pr[A] = 0.05$, $\Pr[B|A] = 0.9$, $\Pr[B|\bar{A}] = 0.2$.

What is the probability of $A$ given that $B$ occurred, $\Pr[A|B]$?
Useful conditional probability facts

Let $A$ be an event and let $E_1, \ldots, E_n$ be mutually disjoint events whose union is $\Omega$.

- **Total probability law.**
  \[
  \Pr[A] = \sum_{i \in [n]} \Pr[A \cap E_i] = \sum_{i \in [n]} \Pr[A \mid E_i] \cdot \Pr[E_i].
  \]

- **Bayes’ Law.** Assume $\Pr[A] \neq 0$. Then
  \[
  \Pr[E_i \mid A] = \frac{\Pr[E_i \cap A]}{\Pr[A]}
  = \frac{\Pr[E_i \mid A] \cdot \Pr[A]}{\sum_{i \in [n]} \Pr[A \mid E_i] \cdot \Pr[E_i]}.
  \]