Randomness in Computing

LECTURE 3

CS 537

Reminders

- Office hours 12:30-1:30 today
- HW2 due Thursday
- PhD students: sign up to grade

Last time

- Probability amplification
- Verifying matrix multiplication
- Probability amplification: Bayesian approach

Today

- Randomized min-cut algorithm

HW guidelines

For every probability question:

- **Define** the events you are considering
- **Justify** your calculations by naming probability laws (or definitions) you are using.
- Mention special properties you are using: for example, independence.

Review question: card dealing

We deal two cards.

Let A_i be the event that the i-th card is an ace for $i \in \{0,1\}$.

What is the probability of A_2?

A. 3/52

B. 3/51

C. 4/52

D. 5/52

E. None of the answers above are correct.

Are A_1 and A_2 independent?

Review question: tennis match

You will play a tennis match against opponent X or Y.

If X is chosen, you win with probability 0.7.

If Y is chosen, you win with probability 0.3.

Your opponent is chosen by flipping a coin with bias 0.6 in favor of X.

What is your probability of winning?

A. < 0.3

B. In the interval $[0.3,0.4)$.

C. In the interval $[0.4,0.55)$.

D. In the interval $[0.55,0.7)$.

E. ≥ 0.7

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Review question: balls and bins

We have two bins with balls.

- Bin 1 contains 3 black balls and 2 white balls.
- Bin 2 contains 1 black ball and 1 white ball.

We pick a bin uniformly at random. Then we pick a ball uniformly at random from that bin.

Find the probability we picked bin 1, given that we picked a white ball.

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§1.5 (MU) Randomized Min Cut

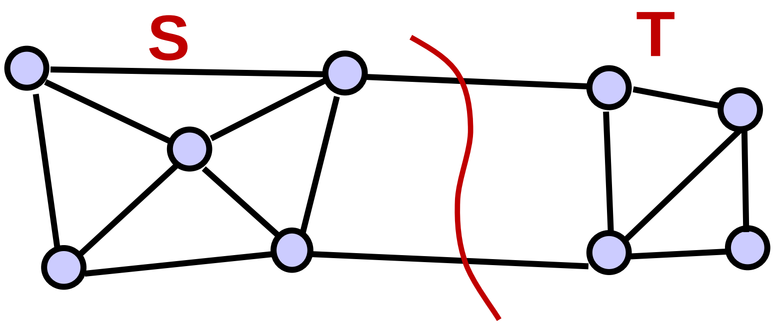
Given: undirected graph $G = (V, E)$

A ***global cut*** of G is a partition of V into non-empty, disjoint sets S, T.

The ***cutset*** of the cut is the set of edges that connect the parts:

$$\{\{u, v\} | u \in S, v \in T\}$$

Goal: Find the min cut in G (a cut with the smallest cutset).



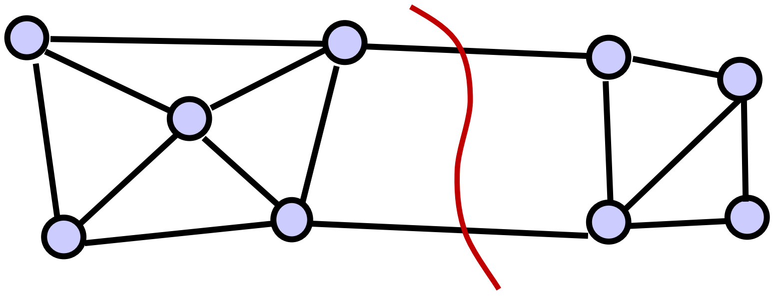
Applications: Network reliability, network design, clustering

Exercise: How many distinct cuts are there in a graph G with n nodes?

Min Cut Algorithms

Given: undirected graph $G = (V, E)$ with n nodes and m edges.

Goal: Find the min cut in G.



Algorithms for Min Cut:

- Deterministic [Stoer-Wagner `97] $O(mn + n^2 \log n)$ time
- Randomized [Karger `93] $O(n^2 m \log n)$ time

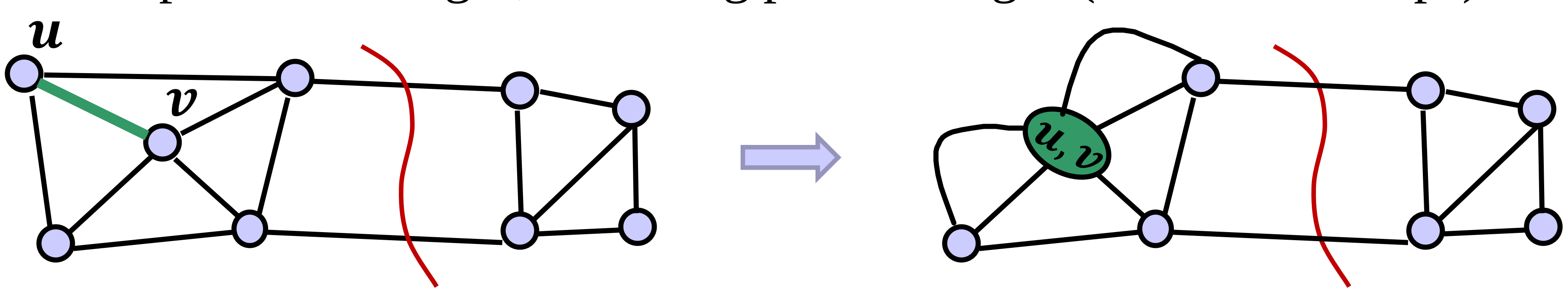
but there are improvements

§1.5 (MU) Karger's Min Cut Algorithm

Idea: Repeatedly pick a random edge and put its endpoints on the same side of the cut.

Basic operation: ***Edge contraction of an edge*** $(\boldsymbol{u}, \boldsymbol{v})$

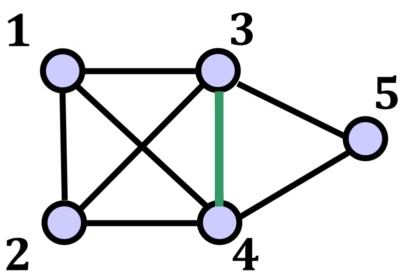
- Merge u and v into one node
- Eliminate all edges connecting u and v
- Keep all other edges, including parallel edges (but no self-loops)

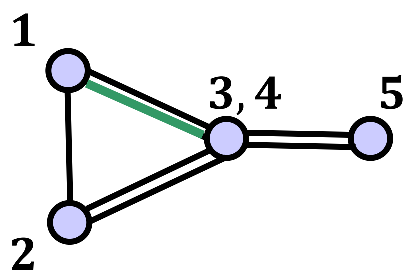


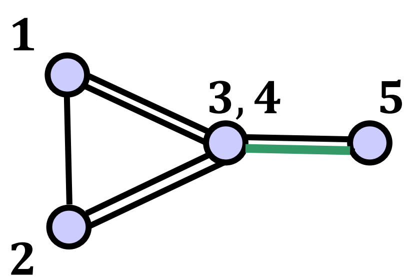
Claim

A cutset of the contracted graph is also a cutset of the original graph.

Example Execution of Karger's Algorithm







§1.5 (MU) Karger's Min Cut Algorithm

Algorithm Basic Karger (input: undirected graph $G = (V, E)$)

1. While $|V| > 2$
2. choose $e \in E$ uniformly at random
3. $G \leftarrow$ graph obtained by contracting e in G
4. **Return** the only cut in G.

Theorem

Basic-Karger returns a min cut with probability $\geq \frac{2}{n(n-1)}$.

Probability amplification for Karger's algorithm

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Basic-Karger returns a min cut with probability $\geq \frac{2}{n(n-1)}$.

Probability Amplification: Repeat $r = n(n-1)\ln n$ times and return the smallest cut found.

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Running time of Basic Karger: Best known implementation: $O(m)$

- Easy: $O(m)$ per contraction, so $O(mn)$
- View as Kruskal's MST algorithm in G with $w(e_i) = \pi(i)$ run until two components are left: $O(m \log n)$