LECTURE 3

Last time
- Probability amplification
- Verifying matrix multiplication

Today
- More probability amplification
- Randomized Min-Cut
- Random variables
Review question: balls and bins

We have two bins with balls.

- Bin 1 contains 3 black balls and 2 white balls.
- Bin 2 contains 1 black ball and 1 white ball.

We pick a bin uniformly at random. Then we pick a ball uniformly at random from that bin.

What is the probability that we picked bin 1, given that we picked a white ball?
How does our confidence increase with the number of trials?

- $C =$ event that identity is correct
- $A =$ event that test accepts

Our analysis of Basic Frievalds:
- $\Pr[A|\bar{C}] \leq 1/2$
- 1-sided error: $\Pr[A|C]=1$

Assumption (initial belief or ``prior''): $\Pr[C] = 1/2$

*By Bayes’ Law*

$$\Pr[C|A] = \frac{\Pr[A|C] \cdot \Pr[C]}{\Pr[A|C] \cdot \Pr[C] + \Pr[A|\bar{C}] \cdot \Pr[\bar{C}]} \geq \frac{1 \cdot \frac{1}{2}}{1 \cdot \frac{1}{2} + \frac{1}{2} \cdot \frac{1}{2}} = \frac{2}{3}$$
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\Pr[C|A] = \frac{\Pr[A|C] \cdot \Pr[C]}{\Pr[A|C] \cdot \Pr[C] + \Pr[A|\bar{C}] \cdot \Pr[\bar{C}]}
\]

\[
\geq \frac{1 \cdot \frac{2}{3}}{1 \cdot \frac{2}{3} + \frac{1}{2} \cdot \frac{1}{3}} = \frac{4}{5}
\]
How does our confidence increase with the number of trials?

- \( C \) = event that identity is correct
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Our analysis of Basic Frievalds:

- \( \Pr[A|\bar{C}] \geq 1/2 \)
- 1-sided error: \( \Pr[A|C] = 1 \)

Assumption (initial belief or ``prior''): \( \Pr[C] = \frac{2^i}{2^i + 1} \)

**By Bayes’ Law**

\[
\Pr[C|A] = \frac{\Pr[A|C] \cdot \Pr[C]}{\Pr[A|C] \cdot \Pr[C] + \Pr[A|\bar{C}] \cdot \Pr[\bar{C}]}
\]

\[
\leq \frac{1 \cdot \frac{2^i}{2^i + 1}}{1 \cdot \frac{2^i}{2^i + 1} + \frac{1}{2} \cdot \frac{1}{2^i + 1}} = \frac{2^{i+1}}{2^{i+1} + 1}
\]
Given: undirected graph \( G = (V, E) \)

A **global cut** of \( G \) is a partition of \( V \) into non-empty, disjoint sets \( S, T \). The **cutset** of the cut is the set of edges that connect the parts:

\[ \{(u, v) | u \in S, v \in T\} \]

Goal: Find the min cut in \( G \) (a cut with the smallest cutset).

Applications: Network reliability, network design, clustering

Exercise: How many distinct cuts are there in a graph \( G \) with \( n \) nodes?
Given: undirected graph $G = (V, E)$ with $n$ nodes and $m$ edges.
Goal: Find the min cut in $G$.

Algorithms for Min Cut:
- Deterministic [Stoer-Wagner `97] $O(mn + n^2 \log n)$ time
- Randomized [Karger `93] $O(n^2 m \log n)$ time

but there are improvements
Idea: Repeatedly pick a random edge and put its endpoints on the same side of the cut.

Basic operation: Edge contraction of an edge \((u, v)\)

- Merge \(u\) and \(v\) into one node
- Eliminate all edges connecting \(u\) and \(v\)
- Keep all other edges, including parallel edges (but no self-loops)

Claim

A cutset of the contracted graph is also a cutset of the original graph.
§1.5 (MU) Karger’s Min Cut Algorithm

<table>
<thead>
<tr>
<th>Algorithm Basic Karger (input: undirected graph $G = (V, E)$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1. While $</td>
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<tr>
<td>2. choose $e \in E$ uniformly at random</td>
</tr>
<tr>
<td>3. $G \leftarrow$ graph obtained by contracting $e$ in $G$</td>
</tr>
<tr>
<td>4. Return the only cut in $G$.</td>
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**Theorem**

Basic-Karger returns a min cut with probability \( \geq \frac{2}{n(n-1)} \).

**Probability Amplification:** Repeat $r = n(n - 1) \ln n$ times and return the smallest cut found.

**Running time of Basic Karger:** Best known implementation: $O(m)$
- Easy: $O(m)$ per contraction, so $O(mn)$
- View as Kruskal’s MST algorithm in $G$ with $w(e_i) = \pi(i)$ run until two components are left: $O(m \log n)$
Measurements in random experiments

- **Example 1:** coin flips
  - Measurement $X$: number of heads.
  - E.g., if the outcome is HHTH, then $X=3$.

- **Example 2:** permutations
  - $n$ students exchange their hats, so that everybody gets a random hat
  - Measurement $X$: number of students that got their own hats.
  - E.g., if students 1,2,3 got hats 2,1,3 then $X=1$. 
Random variables: definition

• A **random variable** $X$ on a sample space $\Omega$ is a function $X : \Omega \rightarrow \mathbb{R}$ that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.

• For each random variable, we should understand:
  – The set of values it can take.
  – The probabilities with which it takes on these values.

• The **distribution** of a discrete random variable $X$ is the collection of pairs $\{(a, \Pr[X = a])\}$. 