LECTURE 4

Last time
• Randomized min-cut algorithm
• Amplification: Bayesian approach

Today
• Finish Karger’s algorithm
• Random variables
• Expectation
• Linearity of expectation
• Jensen’s inequality
§1.5 (MU) Karger’s Min Cut Algorithm

Algorithm Basic Karger (input: undirected graph $G = (V, E)$)

1. While $|V| > 2$
2. choose $e \in E$ uniformly at random
3. $G \leftarrow$ graph obtained by contracting $e$ in $G$
4. **Return** the only cut in $G$.

Theorem

Basic-Karger returns a min cut with probability $\geq \frac{2}{n(n-1)}$. 
Probability amplification for Karger’s algorithm

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**Probability Amplification:** Repeat \( r = n(n - 1) \ln n \) times and return the smallest cut found.
Karger’s Min Cut Algorithm

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**Probability Amplification:** Repeat $r = n(n - 1) \ln n$ times and return the smallest cut found.

**Running time of Basic Karger:** Best known implementation: $O(m)$

- Easy: $O(m)$ per contraction, so $O(mn)$
- View as Kruskal’s MST algorithm in $G$ with $w(e_i) = \pi(i)$ run until two components are left: $O(m \log n)$
• **Example 1:** coin flips
  – Measurement X: number of heads.
  – E.g., if the outcome is HHTH, then $X=3$.

• **Example 2:** permutations
  – $n$ students exchange their hats, so that everybody gets a random hat
  – Measurement X: number of students that got their own hats.
  – E.g., if students 1,2,3 got hats 2,1,3 then $X=1$. 

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A random variable $X$ on a sample space $\Omega$ is a function $X: \Omega \rightarrow \mathbb{R}$ that assigns to each sample point $\omega \in \Omega$ a real number $X(\omega)$.

For each random variable, we should understand:

- The set of values it can take.
- The probabilities with which it takes on these values.

The distribution of a discrete random variable $X$ is the collection of pairs $\{(a, \Pr[X = a])\}$. 
You roll two dice. Let $X$ be the random variable that represents the sum of the numbers you roll.

What is the probability of the event $X=6$?

A. $1/36$
B. $1/9$
C. $5/36$
D. $1/6$
E. None of the above.
You roll two dice. Let $X$ be the random variable that represents the sum of the numbers you roll.

How many different values can $X$ take on?

A. 6  
B. 11  
C. 12  
D. 36  
E. None of the above.
You roll two dice. Let $X$ be the random variable that represents the sum of the numbers you roll.

What is the distribution of $X$?

A. Uniform distribution on the set of possible values.
B. It satisfies $\Pr[X = 2] < \Pr[X = 3] < \cdots < \Pr[X = 12]$.
C. It satisfies $\Pr[X = 2] > \Pr[X = 3] > \cdots > \Pr[X = 12]$.
D. It satisfies $\Pr[X = 2] < \Pr[X = 3] < \cdots < \Pr[X = 7]$ and $\Pr[X = 7] > \Pr[X = 8] > \cdots > \Pr[X = 12]$.
E. None of the above is true.
Independent RVs: definition

- Random variables $X$ and $Y$ are **independent** if
  \[
  \Pr[(X = x) \cap (Y = y)] = \Pr[X = x] \cdot \Pr[Y = y]
  \]
  for all values $x$ and $y$.

- Random variables $X_1, X_2, \ldots, X_n$ are **mutually independent** if for all subsets of $I \subseteq [n]$ and all values $x_i$, where $i \in I$,
  \[
  \Pr[\bigcap_{i \in I} (X_i = x_i)] = \prod_{i \in I} \Pr[X_i = x_i].
  \]
You roll one die. Let $X$ be the random variable that represents the result.

What value does $X$ take, on average?

A. $\frac{1}{6}$
B. 3
C. 3.5
D. 6
E. None of the above.
Random variables: expectation

- The **expectation** of a discrete random variable $X$ over a sample space $\Omega$ is
  \[ \mathbb{E}[X] = \sum_{\omega \in \Omega} X(\omega) \cdot \Pr[\omega]. \]
- We can group together outcomes $\omega$ for which $X(\omega) = a$:
  \[ \mathbb{E}[X] = \sum_a a \cdot \Pr[X = a], \]
  where the sum is over all possible values $a$ taken by $X$.
- The second equality is more useful for calculations.
• **Example:** permutations
  – $n$ students exchange their hats, so that everybody gets a random hat
  – R.V. $X$: the number of students that got their own hats.
  – E.g., if students 1,2,3 got hats 2,1,3 then $X=1$.

• Distribution of $X$:
  \[
  \text{Pr}[X = 0] = \frac{1}{3}, \text{Pr}[X = 1] = \frac{1}{2}, \text{Pr}[X = 3] = \frac{1}{6}.
  \]

• What’s the expectation of $X$?
• **Theorem.** For any two random variables $X$ and $Y$ on the same probability space,

$$\mathbb{E}[X + Y] = \mathbb{E}[X] + \mathbb{E}[Y].$$

Also, for all $c \in \mathbb{R},$

$$\mathbb{E}[cX] = c \cdot \mathbb{E}[X].$$
• An indicator random variable takes on two values: 0 and 1.

• Lemma. For an indicator random variable $X$, $\mathbb{E}[X] = \operatorname{Pr}[X = 1]$. 
You have a coin with bias 3/4 (the bias is the probability of HEADS). Let $X$ be the number of HEADS in 1000 tosses of your coin. You represent $X$ as the sum: $X = X_1 + X_2 + \cdots + X_{1000}$.

What is $X_1$?

B. The number of HEADS.
C. The probability of HEADS in toss 1.
D. The number of heads in toss 1.
E. None of the above.
You have a coin with bias 3/4 (the bias is the probability of HEADS). Let X be the number of HEADS in 1000 tosses of your coin.

What is the expectation of X?

B. 4/3.
C. 500.
D. 750.
E. None of the above.
Example: random hats

• **Example:** permutations
  – $n$ students exchange their hats, so that everybody gets a random hat
  – R.V. $X$: the number of students that got their own hats.
  – E.g., if students 1,2,3 got hats 2,1,3 then $X=1$.

• What’s the expectation of $X$ for general $n$?
Jensen’s inequality: example

- **Exercise:** Let $X$ be the length of a side of a square chosen from $[99]$ uniformly at random. What is the expected value of the area?

  **Solution:** Find $\mathbb{E}[X^2]$.

  $\mathbb{E}[X^2] = $

- **Comparison.** $(\mathbb{E}[X])^2 = $

- In general, $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$
Jensen’s inequality

In general, $\mathbb{E}[X^2] \geq (\mathbb{E}[X])^2$

Proof: Let $\mu = \mathbb{E}[X]$. Consider $Y = (X - \mu)^2$. 
Jensen’s inequality

• A function \( f : \mathbb{R} \to \mathbb{R} \) is convex if, for all \( x, y \) and all \( \lambda \in [0,1] \),
\[
f(\lambda x + (1 - \lambda)y) \leq \lambda f(x) + (1 - \lambda)f(y).
\]

• Jensen’s inequality. If \( f \) is a convex function and \( X \) is a random variable, then
\[
\mathbb{E}[f(X)] \geq f(\mathbb{E}[X]).
\]
Bernoulli random variables

- A Bernoulli random variable with parameters $p$:
  \[
  \begin{cases}
    1 & \text{with probability } p; \\
    0 & \text{otherwise}.
  \end{cases}
  \]

- The expectation of a Bernoulli R.V. $X$ is
  \[\mathbb{E}[X] = p \cdot 1 + (1 - p) \cdot 0 = p.\]
Binomial random variables

- A **binomial random variable** with parameters $n$ and $p$, denoted $\text{Bin}(n, p)$, is the number of HEADS in $n$ tosses of a coin with bias $p$.

- **Lemma.** The probability distribution of $X = \text{Bin}(n, p)$ is

$$\Pr[X = j] = \binom{n}{j} p^j (1 - p)^{n-j}$$

for all $j = 0, 1, \ldots, n$.

- **Lemma.** The expectation of $X = \text{Bin}(n, p)$ is

$$\mathbb{E}[X] = np.$$
Throw $m$ balls into $n$ bins.

Let $X$ be the number of balls that land into bin 1.

(Recall that $\text{Bin}(n,p)$ is the binomial distribution, i.e., the distribution of the number of HEADS in $n$ tosses of a coin with bias $p$.)

Then the distribution of $X$ is

A. $\text{Bin } (n, m)$
B. $\text{Bin } (m, 1/n)$
C. $\text{Bin } \left( m, \frac{n-1}{n} \right)$
D. a binomial distribution, but none of the above.
E. not a binomial distribution.
Throw $m$ balls into $n$ bins. Let $Y$ be the number of empty bins.

Compute $\mathbb{E}[Y]$.

A. 1
B. $n/2$
C. $\left(1 - \frac{1}{n}\right)^m$
D. $n \left(1 - \frac{1}{n}\right)^m$

E. None of the above.
Product of independent RVs

• **Theorem.** For any two independent random variables $X$ and $Y$ on the same probability space,

$$\mathbb{E}[X \cdot Y] = \mathbb{E}[X] \cdot \mathbb{E}[Y].$$

• **Note.** The equality does not hold, in general, for dependent random variables.

**Example.** We toss two coins.
Let $X=\text{number of HEADS}$, $Y=\text{number of TAILS}$.
Calculate $\mathbb{E}[X]$, $\mathbb{E}[Y]$ and $\mathbb{E}[XY]$. 

Sofya Raskhodnikova; Randomness in Computing