Randomness in Computing

CS 537

Lecture 7

Last time

- Coupon collector problem
- Randomized quicksort

Today

- Markov's inequality
- Variance, covariance

Markov Inequality

Theorem (Markov Inequality)

Let X be a RV taking only nonnegative values. Then, for all $a > 0$,

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

Andrei Markov
[1856-1922]

Proof: Let $a > 0$.

Let I be the indicator R.V. for $X \geq a$: $I = \begin{cases} 1 & \text{if } X \geq a \\ 0 & \text{otherwise} \end{cases}$

Since $X \geq 0$ and $a > 0$,

$$I \leq \frac{X}{a} \qquad (*)$$

image source
https://en.wikipedia.org/wiki/Andrey_Markov

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Alternative roof: Let $a > 0$.

Let A be the event that $X \geq a$.

$\mathbb{E}[X] =$

image source https://en.wikipedia.org/wiki/Andrey_Markov

Markov Inequality: Corollary

Theorem (Markov Inequality)

Let X be a RV taking only nonnegative values.
Then, for all $a > 0$,

$$\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.$$

Alternative form: Then, for all $b > 1$,

$$\Pr[X \geq b \cdot \mathbb{E}(X)] \leq \frac{1}{b}.$$

Andrei Markov
[1856-1922]

Example: Show that Randomized Quicksort uses

$$\leq 10n \ln n + O(n)$$

comparisons with probability at least 4/5.

image source
https://en.wikipedia.org/wiki/Andrey_Markov

Bernoulli random walk

- Start at position 0.
- At every step go up or down by 1 with probability 1/2 each.
- Let X be the position after n steps.

What is the probability space?

A. Uniform over {0,1}.

B. Uniform over {-1,1}.

C. 2^n.

D. Position after n steps.

E. Uniform over $\{(s_1, \ldots, s_n) \mid s_i \in \{-1,1\}$ for $i = 1, \ldots, n\}$.

Bernoulli random walk

- Start at position 0.
- At every step go up or down by 1 with probability 1/2 each.
- Let X be the position after n steps.
- What is $\mathbb{E}[X]$?
- How far from the origin should we expect X to be?

A precise answer to this question is the expectation of $|X|$.
However, it is easier to work with the expectation of X^2.
(It is not the same! But gives us an idea.)

Random variables: variance

- The variance of a random variable X with expectation $\mathbb{E}[X] = \mu$ is

$$\mathrm{Var}[X] = \mathbb{E}[(X-\mu)^2].$$

- Equivalently, $\mathrm{Var}[X] = \mathbb{E}[X^2] - \mu^2$.

- The standard deviation of X is $\sigma[X] = \sqrt{\mathrm{Var}[X]}$.

Variance as a measure of spread

- $X = \begin{cases} -2 \text{ with probability } 1/2 \\ 2 \text{ with probability } 1/2 \end{cases}$
- $Y = \begin{cases} -10 \text{ with probability } 0.001 \\ 0 \text{ with probability } 0.998 \\ 10 \text{ with probability } 0.001 \end{cases}$
- $Z = \begin{cases} -5 \text{ with probability } 1/3 \\ 0 \text{ with probability } 1/3 \\ 5 \text{ with probability } 1/3 \end{cases}$
- Compute the variances and standard deviations of X,Y and Z.

Example from Mathematics: a Discrete introduction by E. Scheinerman

Compute expectation and variance

- Fair die. Let X be the number showing on a roll of a die.

$$\mathrm{Var}[X] = E[X^2] - \mu^2$$

$$= \frac{1+4+9+16+25+36}{6} - \left(\frac{7}{2}\right)^2 = \frac{91}{6} - \frac{49}{4} = \frac{35}{12}$$

- Uniform distribution. X is uniformly distributed over $[n]$.
  - The sum of the first n squares is $\frac{n(n+1)(2n+1)}{6}$

$$\mathrm{Var}[X] = \frac{1}{n}\sum_{i\in[n]} i^2 - \left(\frac{1}{n}\sum_{i\in[n]} i\right)^2$$

$$= \frac{(n+1)(2n+1)}{6} - \frac{(n+1)^2}{4} = \frac{n^2-1}{12}$$

Compute expectation and variance

Number of fixed points of a permutation. Let X be the number of students that get their hats back when n students randomly switch hats, so that every permutation of hats is equally likely.

Solution: X_i= the indicator R.V. for person i getting their hat back.

$$X = X_1 + \cdots + X_n$$

By linearity of expectation and symmetry, $\mathbb{E}[X] = n \cdot \mathbb{E}[X_1] = n \cdot \frac{1}{n} = 1$

$$\begin{aligned}\mathbb{E}[X^2] &= \mathbb{E}[(X_1 + \cdots + X_n)^2] \\ &= n \cdot \mathbb{E}[X_1^2] + n(n-1) \cdot \mathbb{E}[X_1 \cdot X_2] \\ &= n \cdot \frac{1}{n} + n(n-1) \cdot \frac{1}{n(n-1)} = 2\end{aligned}$$

$$\mathrm{Var}[X] = \mathbb{E}[X^2] - (\mathbb{E}[X])^2 = 2 - 1 = 1$$

Random variables: covariance

- The covariance of two random variables X and Y with expectations $\mathbb{E}[X] = \mu_X$ and $\mathbb{E}[Y] = \mu_Y$ is
$$\mathrm{Cov}(X, Y) = \mathbb{E}[(X - \mu_X)(Y - \mu_Y)].$$

- Theorem. For any two random variables X and Y,
$$\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y] + 2\,\mathrm{Cov}(X, Y).$$
- Proof: $\mathrm{Var}[X + Y]$
$$= \mathbb{E}\left[\big((X + Y) - \mathbb{E}[X + Y]\big)^2\right]$$
$$= \mathbb{E}[((X - \mu_X) + (Y - \mu_Y))^2]$$
$$= \mathbb{E}[(X - \mu_X)^2 \quad + \quad (Y - \mu_Y)^2 \quad + \quad 2(X - \mu_X)(Y - \mu_Y)]$$
$$= \mathbb{E}[(X - \mu_X)^2] + \mathbb{E}[(Y - \mu_Y)^2] + 2\mathbb{E}[(X - \mu_X)(Y - \mu_Y)]$$
$$= \qquad \mathrm{Var}[X] \qquad + \qquad \mathrm{Var}[Y] \qquad + \qquad 2\,\mathrm{Cov}(X, Y)$$

Independent RVs

- Random variables X and Y on the same probability space are independent if for all values a and b, the events $X = a$ and $Y = b$ are independent.
  Equivalently, for all a, b,
  $$\Pr[X = a \wedge Y = b] = \Pr[X = a] \cdot \Pr[Y = b].$$
- Theorem. For independent random variables X and Y,
  1) $\mathbb{E}[XY] = \mathbb{E}[X] \cdot \mathbb{E}[Y]$.
  2) $\mathrm{Var}[X + Y] = \mathrm{Var}[X] + \mathrm{Var}[Y]$.
  3) Cov(X,Y)=0.

Example: n coin tosses

- Let X be the number of HEADS in n tosses of a biased coin with HEADS probability p.
- We know: X has binomial distribution $\text{Bin}(n, p)$.
- What is the variance of X?

Answer: $np(1 - p)$.

Example: Geometric RV

- Let X be the # of coin tosses until the first HEADS of a biased coin with HEADS probability p.
- We know: X has geometric distribution Geom(p).
- What is the variance of X?

Answer: $\frac{1-p}{p^2}$.