Lecture 10

Last time
- Median of a RV
- Computing the median of an array

Today
- Chernoff Bounds
• **Markov.** For a nonnegative random variable $X$ and $a > 0$,

\[
\Pr[X \geq a] \leq \frac{\mathbb{E}[X]}{a}.
\]

• **Chebyshev.** For a random variable $X$ and $a > 0$,

\[
\Pr[|X - \mathbb{E}[X]| \geq a] \leq \frac{\text{Var}[X]}{a^2}.
\]
Chernoff Bound (Upper Tail). Let $X_1, \ldots, X_n$ be independent Bernoulli RVs. Let $X = X_1 + \cdots + X_n$ and $\mu = E[X]$. Then

- (stronger) for any $\delta > 0$,
  \[
  \Pr[X \geq (1 + \delta)\mu] \leq \left(\frac{e^{\delta}}{(1 + \delta)^{1+\delta}}\right)^\mu.
  \]

- (easier to use) for any $\delta \in (0,1]$,
  \[
  \Pr[X \geq (1 + \delta)\mu] \leq e^{-\mu\delta^2/3}.
  \]
C Chernoff Bound (Lower Tail). Let \( X_1, \ldots, X_n \) be independent Bernoulli RVs. Let \( X = X_1 + \cdots + X_n \) and \( \mu = E[X] \). Then

- (stronger) for any \( \delta \in (0,1) \),

\[
\Pr[X \leq (1 - \delta)\mu] \leq \left( \frac{e^{-\delta}}{(1 - \delta)^{1-\delta}} \right)^\mu.
\]

- (easier to use) for any \( \delta \in (0,1) \),

\[
\Pr[X \leq (1 - \delta)\mu] \leq e^{-\mu\delta^2/2}.
\]
Sums of independent RVs

Chernoff Bound (Both Tails). Let $X_1, \ldots, X_n$ be independent Bernoulli RVs. Let $X = X_1 + \cdots + X_n$ and $\mu = E[X]$. Then

- for any $\delta \in (0,1)$,

$$
\Pr[|X - \mu| \geq \delta \mu] \leq 2e^{-\mu \delta^2 / 3}.
$$
The Halting Problem Team wins each hockey game they play with probability 1/3. Assuming outcomes of the games are independent, derive an upper bound on the probability that they have a winning season in \( n \) games.

The Halting Problem Team hires a new coach, and critics revise their probability of winning each game to 3/4. Derive an upper bound on the probability they suffer a losing season.
Exercise

• We throw $n$ balls uniformly and independently into $n$ bins.
Let $Y_1$ be the number of balls that fell into bin 1.
Determine $m$ such that $\Pr[Y_1 > m] \leq \frac{1}{n^2}$. 
Tail bounds so far

- **Markov.** For a nonnegative random variable $X$ and $a > 0$,
  \[ \Pr[X \geq a] \leq \frac{E[X]}{a}. \]

- **Chebyshev.** For a random variable $X$ and $a > 0$,
  \[ \Pr[|X - E[X]| \geq a] \leq \frac{\text{Var}[X]}{a^2}. \]

- **Example 1:** $X \sim \text{Bin}(n, 1/2)$.
  Bound $\Pr[X > \frac{3n}{4}]$ using Markov and Chebyshev.

- **Example 2:** Coupon Collector Problem.
  Bound $\Pr[X > 2nH_n]$ using Markov and Chebyshev.