Randomness in Computing

CS 537

LECTURE 10

Last time

- Computing the median of an array
- Chernoff Bounds

Today

- Chernoff Bounds
- Hoeffding Bounds
- Applications of Chernoff-Hoeffding Bounds

Sums of Independent Bernoullis

Chernoff Bound (Upper Tail)

Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.
Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (stronger) for all $\delta > 0$,

$$\Pr[X \geq (1+\delta)\mu] \leq \left(\frac{e^{\delta}}{(1+\delta)^{1+\delta}}\right)^{\mu}.$$

- (easier to use) for all $\delta \in (0,1]$,

$$\Pr[X \geq (1+\delta)\mu] \leq e^{-\mu\delta^2/3}.$$

Sums of Independent Bernoullis

Chernoff Bound (Lower Tail)

Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.
Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (stronger) for all $\delta \in (0,1)$,

$$\Pr[X \le (1-\delta)\mu] \le \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

- (easier to use) for all $\delta \in (0,1)$,

$$\Pr[X \le (1-\delta)\mu] \le e^{-\mu\delta^2/2}.$$

Proof sketch for lower tail

- For all real $t < 0$,

$$\Pr[X \le (1-\delta)\mu] = \Pr\left[e^{tX} \ge e^{t(1-\delta)\mu}\right] \le \frac{\mathbb{E}[e^{tX}]}{e^{t(1-\delta)\mu}}$$

by Markov

$$\le \left[\frac{e^{e^{t}-1}}{e^{t(1-\delta)}}\right]^{\mu}$$

Setting the value of t:

$$\boxed{e^{t} = 1-\delta}$$

Sums of Independent Bernoullis

Chernoff Bound (Lower Tail)

Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.
Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (stronger) for all $\delta \in (0,1)$,

$$\Pr[X \le (1-\delta)\mu] \le \left(\frac{e^{-\delta}}{(1-\delta)^{1-\delta}}\right)^{\mu}.$$

- (easier to use) for all $\delta \in (0,1)$,

$$\Pr[X \le (1-\delta)\mu] \le e^{-\mu\delta^2/2}.$$

Sums of Independent Bernoullis

Chernoff Bound (Both Tails)

Let $X_1, \ldots, X_n$ be independent Bernoulli RVs.
Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- for all $\delta \in (0,1)$,

$$\Pr[|X - \mu| \geq \delta\mu] \leq 2e^{-\mu\delta^2/3}.$$

Exercise

$$X \sim \mathrm{Bin}\left(n, \frac{1}{2}\right) \qquad \Pr\left[X \geq \frac{3n}{4}\right] \leq ?$$

- Recall: $\mathbb{E}[X] = \qquad , \mathrm{Var}[X] =$
- Markov: $\Pr[X \geq 3n/4] \leq$
- Chebyshev: $\leq 2/n$
- Chernoff: $\Pr[X \geq \qquad] \leq e^{-}$

$$\Pr\left[X \leq \frac{n}{4}\right] \leq$$

$$\Pr\left[X \leq \frac{n}{2} - c\sqrt{n}\right] = \Pr\left[X \leq (1 - \qquad) \frac{n}{2}\right] \leq e^{-}$$

Exercise 2

- The Halting Problem Team wins each hockey game they play with probability 1/3. Assuming outcomes of the games are independent, derive an upper bound on the probability that they have a winning season in n games.

- The Halting Problem Team hires a new coach, and critics revise their probability of winning each game to 3/4. Derive an upper bound on the probability they suffer a losing season.

Exercise 3

We throw n balls uniformly and independently into n bins.

Let Y_1 be the number of balls that fell into bin 1.

Determine m such that $\Pr[Y_1 > m] \leq \frac{1}{n^2}$.

Sums of Independent Bounded RVs

Hoeffding Bound

Let $X_1, \ldots, X_n$ be independent random variables with $\Pr[a \leq X_i \leq b] = 1$.

Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (upper tail) $\Pr[X \geq \mu + \epsilon n] \leq e^{-2n\epsilon^2/(b-a)^2}$
- (lower tail) $\Pr[X \leq \mu - \epsilon n] \leq e^{-2n\epsilon^2/(b-a)^2}$

Application: Estimating a parameter

- **Unknown:** probability p that a feature occurs in the population.
- Obtain an estimate by taking n samples
- $X \sim \text{Bin}(n, p)$
- Let $\tilde{p} = X/n$.
- A $1 - \gamma$ confidence interval for parameter p is an interval $[\tilde{p} - \epsilon, \tilde{p} + \epsilon]$ such that $\Pr[p \in [\tilde{p} - \epsilon, \tilde{p} + \epsilon]] \geq 1 - \gamma$.
- Find a tradeoff between γ, ϵ and n.

Application: Estimating a parameter

- A $1-\gamma$ confidence interval for parameter p is an interval $[\tilde{p}-\epsilon, \tilde{p}+\epsilon]$ such that $\Pr[p \in [\tilde{p}-\epsilon, \tilde{p}+\epsilon]] \geq 1-\gamma$.
- Find a tradeoff between γ, ϵ and n.

Solution: $\mathbb{E}[X] = np$

- Suppose $p \notin [\tilde{p}-\epsilon, \tilde{p}+\epsilon]$
- Case 1: $p < \tilde{p}-\epsilon$. Then $\tilde{p} > \quad p+\epsilon$
- Case 2: $p > \tilde{p}+\epsilon$. Then $\tilde{p} < \quad p-\epsilon$
- $\gamma = 2 \cdot e^{-2\epsilon^2 n}$