LECTURE 11

Last time
- Chernoff Bounds

Today
- Hoeffding Bounds
- Applications of Chernoff-Hoeffding Bounds
  - Estimating a Parameter
  - Set Balancing
  - Routing on the hypercube
Sums of Independent Bounded RVs

Hoeffding Bound

Let $X_1, \ldots, X_n$ be independent random variables with $\Pr[a \leq X_i \leq b] = 1$. Let $X = X_1 + \cdots + X_n$ and $\mu = \mathbb{E}[X]$. Then

- (upper tail) $\Pr[X \geq \mu + \epsilon n] \leq e^{-2n\epsilon^2/(b-a)^2}$
- (lower tail) $\Pr[X \leq \mu - \epsilon n] \leq e^{-2n\epsilon^2/(b-a)^2}$
Application: Estimating a parameter

- **Unknown**: probability $p$ that a feature occurs in the population.
- Obtain an estimate by taking $n$ samples
- $X \sim \text{Bin}(n, p)$
- Let $\hat{p} = X/n$.
- A $1 - \gamma$ confidence interval for parameter $p$ is an interval $[\hat{p} - \epsilon, \hat{p} + \epsilon]$ such that $\Pr[p \in [\hat{p} - \epsilon, \hat{p} + \epsilon]] \geq 1 - \gamma$.
- Find a tradeoff between $\gamma, \epsilon$ and $n$. 
• A $1 - \gamma$ confidence interval for parameter $p$ is an interval $[\hat{p} - \epsilon, \hat{p} + \epsilon]$ such that $\Pr[p \in [\hat{p} - \epsilon, \hat{p} + \epsilon]] \geq 1 - \gamma$.

• Find a tradeoff between $\gamma, \epsilon$ and $n$.

**Solution:** $\mathbb{E}[X] = np$

• Suppose $p \not\in [\hat{p} - \epsilon, \hat{p} + \epsilon]$

• Case 1: $p < \hat{p} - \epsilon$. Then $\hat{p} > p + \epsilon$

• Case 2: $p > \hat{p} + \epsilon$. Then $\hat{p} < p - \epsilon$

• $\gamma = 2 \cdot e^{-2\epsilon^2 n}$
**Application: Set Balancing**

- **Given:** an $n \times m$ matrix $A$ with 0-1 entries
- **Definition:** $||(x_1, \ldots, x_n)||_\infty = \max_{i \in [n]} |x_i|$
- **Find:** $b \in \{-1,1\}^m$ minimizing $||Ab||_\infty$

Partition subjects into two groups, so that each feature is balanced.
**Application: Set Balancing**

**Algorithm:** Choose each $b_i$ independently from $\{-1,1\}$.

**Theorem.** $\Pr\left[\|Ab\|_\infty \geq \sqrt{4m \ln n}\right] \leq 2/n$.

**Proof:** Let $\bar{a}_i$ be the vector in row $i$ of $A$ for all $i \in [n]$.

- Let $B_i$ be the (bad) event that $\bar{a}_i \cdot b \geq \sqrt{4m \ln n}$.
- For $i \in [n]$, let $Z_i$ be the random variable $\sum_{j \in [m]} a_{ij}b_j$.
- $\mathbb{E}[Z_i] = \quad$ [m independent RVs]
- By Hoeffding,
  \[ \Pr[B_i] = \Pr[|Z_i| \geq \epsilon m] \leq 2e^{-\frac{2\epsilon^2 m}{4}} = 2e^{-2 \ln n} \leq 2n^{-2} \]
- By a union bound over all $n$ rows,
  \[ \Pr\left[\|Ab\|_\infty \geq \sqrt{4m \ln n}\right] = \Pr\left[\bigcup_{i \in [n]} B_i\right] \]
  $\epsilon m = \sqrt{4m \ln n}$
  \[ \epsilon^2 = \frac{4 \ln n}{m} \]
An $n$-dimensional hypercube is a directed graph with

- $N = 2^n$ nodes, each indexed by $n$-bit integer
- containing the directed edge $(x, y)$ iff $x$ and $y$ differ in exactly one bit

How many edges?

- **Routing.** Each node is a routing switch. 
  Edges = communication channels 
  An edge can carry one packet in one step.

- A *routing algorithm* specifies a path from $s$ to $t$ for each pair of nodes $(s, t)$ and a *queuing policy* for ordering packets that are waiting for the same link (e.g., FIFO – First In First Out or FTG – Furthest to Go)
Permutation Routing Problem

• Each node is the source of one packet
• Each node is the designation of one packet
  ➢ E.g., on a complete graph, it can be solved in one time step.
  ➢ On sparse graphs?

Hypercube: $N$ nodes $N\log N$ edges.

### Bit-Fixing Routing Algorithm for the Hypercube

1. Let $x$ be the current node and $y$ be the destination of a packet
2. Find smallest $i \in [n]$ such that $x_i \neq y_i$
3. Traverse the edge $(x, x_1 \ldots x_{i-1} \bar{x}_i x_{i+1} \ldots x_n)$

How long does it take to route a packet if there are no delays?
Bad example for congestion

• **Transpose permutation** \((n \text{ is even})\)
  
  From each \(x\), send a packet to \((x_{n/2+1}, \ldots, x_n, x_1, \ldots, x_{n/2})\)

  **Exercise:** Show that Bit-Fixing Algorithm takes \(\Omega(\sqrt{N})\) steps on transpose permutation.

• **Known:**

  Any deterministic *oblivious* algorithm on a network with \(N\) nodes, each of outdegree \(d\), takes \(\Omega\left(\sqrt{N/d}\right)\) steps on some permutation.

  Randomization is essential!
Randomized Routing

**Randomized Routing Algorithm (RRA)**

0. For each packet going from $x$ to $y$, pick a node $z$ uniformly and independently at random.
1. Use Bit-Fixing to route the packet from $x$ to $z$.
2. Use Bit-Fixing to route the packet from $z$ to $y$.

**Theorem**

For every permutation, RRA takes $O(\log N)$ steps w.p. $1 - O\left(\frac{1}{N}\right)$.

**Proof:**

**Idea 1:** Analyze Phase 1. (Phase 2 is ``symmetric”’’ going backwards; first think about waiting with Phase 2 until all packets are done with Phase 1.)

**Idea 2:** In an intermediate destination, each bit $z_i$ is 0/1 uniformly & independently

**Idea 3:** # steps in Phase1 is (# bits to fix) + (waiting time in queues in Phase 1)

\[ \leq n + \text{delay} \]
Lemma

Let \( p_i \) be the path of some packet \( i \) in Phase 1.
Let \( S \) be the set of packets (other than \( i \)) whose routes pass through at least one edge of \( p_i \). Then the delay of \( i \) is at most \( |S| \).
### Main Lemma

Consider any packet $i$. It fails to reach its destination in phase 1 within $3n$ steps with probability at most $\frac{1}{N^2}$.

**Proof:** Let $X$ be the number of packets (other than $i$) that use at least one edge from the path of $i$.

We would like to find an upper bound on $\mathbb{E}[X]$. 

Sofya Raskhodnikova; Randomness in Computing
**Analysis of RRA**

**Main Lemma**
Consider any packet \( i \). It fails to reach its destination in phase 1 within \( 3n \) steps with probability at most \( \frac{1}{N^2} \).

**Proof:** Let \( X \) be the number of packets (other than \( i \)) that use at least one edge from the path of \( i \).

- For any edge \( e \), let \( Y_e = \# \) routes that pass via \( e \).
- Let \( p_i = (e_1, \ldots, e_K) \) be the path of packet \( i \). Then \( K \leq n \).
- Then \( X \leq Y_{e_1} + Y_{e_2} + \cdots + Y_{e_K} \).
- By symmetry of the hypercube, \( \mathbb{E}[Y_e] \) is the same for all edges.
- By linearity of expectation,
  \[
  \mathbb{E}[X|K = k] \leq \mathbb{E}[Y_{e_1}] + \cdots + \mathbb{E}[Y_{e_k}] = k \cdot \mathbb{E}[Y_e]
  \]
Calculating $\mathbb{E}[Y_e]$ 

- For any edge $e$, let $Y_e = \#$ routes that pass via $e$.
- Suppose $e$ is an edge in dimension $d \in [n]$, that is, $e = (x_1 \ldots x_n, x_1 \ldots x_{d-1} \overline{x_d} \ldots x_n)$
- Only packets with source $\ast \ldots \ast x_d \ldots x_n$ can traverse $e$
- To traverse $e$, such a packet must have designation which happens with probability
- Thus, $\mathbb{E}[Y_e] =$
Analysis of RRA

Main Lemma

Consider any packet $i$. It fails to reach its destination in phase 1 within $3n$ steps with probability at most $\frac{1}{N^2}$.

Proof: Let $X$ be the number of packets (other than $i$) that use at least one edge from the path of $i$.

- For any edge $e$, let $Y_e = \#$ routes that pass via $e$.
- Let $p_i = (e_1, ..., e_K)$ be the path of packet $i$. Then $K \leq n$.
- By linearity of expectation,
  $$\mathbb{E}[X|K = k] \leq \mathbb{E}[Y_{e_1}] + \cdots + \mathbb{E}[Y_{e_k}] = k \cdot \mathbb{E}[Y_e] =$$
- By compact law of total expectation, $\mathbb{E}[X]$
  $$= \mathbb{E}[\mathbb{E}[X|K]] \leq \mathbb{E}[K/2] =$$
### Analysis of RRA

#### Main Lemma

Consider any packet $i$. It fails to reach its destination in phase 1 within $3n$ steps with probability at most $\frac{1}{N^2}$.

**Proof:** Let $X$ be the number of packets (other than $i$) that use at least one edge from the path of $i$.

- So far: $\mathbb{E}[X] \leq \frac{n}{4}$ and travel time for packet $i$ is $\leq n + X$
- By Chernoff bound, $\Pr[X \geq 2n] =$
Using the Main Lemma to complete the analysis:

• By a union bound over $N$ packets, the probability that at least one packet fails to reach its destination in phase 1 within $3n$ steps is