LECTURE 12

Last time
• Chernoff Bounds
• Estimating a parameter
• Set Balancing

Today
• Routing on the hypercube
• The Balls-and-Bins model
An $n$-dimensional hypercube is a directed graph with

- $N = 2^n$ nodes, each indexed by $n$-bit integer
- containing the directed edge $(x, y)$ iff $x$ and $y$ differ in exactly one bit

How many edges?

- **Routing.** Each node is a routing switch. Edges = communication channels
  An edge can carry one packet in one step.
- A routing algorithm specifies a path from $s$ to $t$ for each pair of nodes $(s, t)$ and a queuing policy for ordering packets that are waiting for the same link (e.g., FIFO – First In First Out or FTG – Furthest to Go)
Permutation Routing Problem

- Each node is the source of one packet
- Each node is the designation of one packet
  
  - E.g., on a complete graph, it can be solved in one time step.
  - On sparse graphs?

Hypercube: $N$ nodes $N \log N$ edges.

### Bit-Fixing Routing Algorithm for the Hypercube

1. Let $x$ be the current node and $y$ be the destination of a packet
2. Find smallest $i \in [n]$ such that $x_i \neq y_i$
3. Traverse the edge $(x, x_1 \ldots x_{i-1} x_i x_{i+1} \ldots x_n)$

How long does it take to route a packet if there are no delays?
Bad example for congestion

• Transpose permutation \((n \text{ is even})\)
  From each \(x\), send a packet to \((x_{n/2+1}, \ldots, x_n, x_1, \ldots, x_{n/2})\)

HW: Show that Bit-Fixing Algorithm takes \(\Omega(\sqrt{N})\) steps on transpose permutation.

• Known:
  Any deterministic oblivious algorithm on a network with \(N\) nodes, each of outdegree \(d\), takes \(\Omega \left( \sqrt{N/d} \right)\) steps on some permutation.

  Randomization is essential!
Randomized Routing

**Randomized Routing Algorithm (RRA)**

0. For each packet going from $x$ to $y$, pick a node $z$ uniformly and independently at random.
1. Use Bit-Fixing to route the packet from $x$ to $z$.
2. Use Bit-Fixing to route the packet from $z$ to $y$.

**Theorem**

$log N = n$

For every permutation, RRA takes $O(\log N)$ steps w.p. $1 - O\left(\frac{1}{N}\right)$.

**Proof:**

**Idea 1:** Analyze Phase 1. (Phase 2 is ``symmetric”’’ going backwards; first think about waiting with Phase 2 until all packets are done with Phase 1.)

**Idea 2:** In an intermediate destination, each bit $z_i$ is 0/1 uniformly & independently

**Idea 3:** # steps in Phase1 is (# bits to fix) + (waiting time in queues in Phase 1)

$\leq n + \text{delay}$
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<th>Lemma</th>
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| Let $p_i$ be the path of some packet $i$ in Phase 1.  
Let $S$ be the set of packets (other than $i$) whose routes pass through at least one edge of $p_i$. Then the delay of $i$ is at most $|S|$. |
### Main Lemma

Consider any packet $i$. It fails to reach its destination in phase 1 within $3n$ steps with probability at most $\frac{1}{N^2}$.

**Proof:** Let $X$ be the number of packets (other than $i$) that use at least one edge from the path of $i$.

We would like to find an upper bound on $\mathbb{E}[X]$. 
**Main Lemma**

Consider any packet \( i \). It fails to reach its destination in phase 1 within \( 3n \) steps with probability at most \( \frac{1}{N^2} \).

**Proof:** Let \( X \) be the number of packets (other than \( i \)) that use at least one edge from the path of \( i \).

- For any edge \( e \), let \( Y_e = \# \) routes that pass via \( e \).
- Let \( p_i = (e_1, \ldots, e_K) \) be the path of packet \( i \). Then \( K \leq n \).
- Then \( X \leq Y_{e_1} + Y_{e_2} + \cdots + Y_{e_K} \).
- By symmetry of the hypercube, \( \mathbb{E}[Y_e] \) is the same for all edges.
- By linearity of expectation,
  \[
  \mathbb{E}[X | K = k] \leq \mathbb{E}[Y_{e_1}] + \cdots + \mathbb{E}[Y_{e_k}] = k \cdot \mathbb{E}[Y_e]
  \]
Calculating $\mathbb{E}[Y_e]$ 

- For any edge $e$, let $Y_e = \#$ routes that pass via $e$.
- Suppose $e$ is an edge in dimension $d \in [n]$, that is, $e = (x_1 \ldots x_n, x_1 \ldots x_{d-1} \overline{x_d} \ldots x_n)$
- Only packets with source $\ast \ldots \ast x_d \ldots x_n$ can traverse $e$
- To traverse $e$, such a packet must have designation which happens with probability
- Thus, $\mathbb{E}[Y_e] = \ldots$
## Analysis of RRA

### Main Lemma

Consider any packet $i$. It fails to reach its destination in phase 1 within $3n$ steps with probability at most $\frac{1}{N^2}$.

**Proof:** Let $X$ be the number of packets (other than $i$) that use at least one edge from the path of $i$.

- For any edge $e$, let $Y_e = \#$ routes that pass via $e$.
- Let $p_i = (e_1, \ldots, e_K)$ be the path of packet $i$. Then $K \leq n$.
- By linearity of expectation,
  \[
  \mathbb{E}[X|K = k] \leq \mathbb{E}[Y_{e_1}] + \cdots + \mathbb{E}[Y_{e_k}] = k \cdot \mathbb{E}[Y_e] =
  \]
- By compact law of total expectation, $\mathbb{E}[X] = \mathbb{E}[\mathbb{E}[X|K]] \leq \mathbb{E}[K/2] =
  

Find an upper bound on $\mathbb{E}[X]$.
Main Lemma

Consider any packet $i$. It fails to reach its destination in phase 1 within $3n$ steps with probability at most $\frac{1}{N^2}$.

Proof: Let $X$ be the number of packets (other than $i$) that use at least one edge from the path of $i$.

- So far: $\mathbb{E}[X] \leq \frac{n}{4}$ and travel time for packet $i$ is $\leq n + X$

- By Chernoff bound, $\Pr[X \geq 2n] =

- By a union bound over $N$ packets, the probability that at least one packet fails to reach its destination in phase 1 within $3n$ steps is
The Balls-and-Bins Model

• \( m \) balls thrown into \( n \) bins

Each ball falls into a uniformly random bin (u.i.r.)

Q1. Is it more likely that there is a collision or not?
   (Birthday Paradox)

Q2. How many balls are in the fullest bin? (Maximum load)

Q3. How many bins are empty?

Q4. What does the distribution of the balls in the bins look like?
Birthday Paradox

- $n = 365$ bins (days)
- What is the probability that all $m$ people have different birthdays?
- For which $m$ is the probability of collision more than 1/2?

Let $E_i$ for $i \in [m]$ be the event that ball $i$ falls into an empty bin.

$\Pr[\text{no collision}]$