Lecture 24

Last time
- Probabilistic method
  - Algorithmic LLL
  - Applications of LLL

Today
- Drunkard’s walk
- Markov chains
- Randomized algorithm for 2SAT
Drunkard’s walk problem

$p_j = \Pr[\text{Tipsy goes home } | \text{ he started at position } j]$  

$p_n = 1$  

$p_0 = 0$
Drunkard’s walk: probability

\[ p_j = \Pr[\text{Tipsy goes home} \mid \text{he started at position } j] \]

\[ p_n = 1 \]

\[ p_0 = 0 \]
Drunkard’s walk: probability

\[ p_j = \Pr[\text{Tipsy goes home } \mid \text{he started at position } j] \]

\[
p_n = 1 \\
p_0 = 0
\]

for all \( j \in [1, n - 1] \):

\[
p_j = \frac{p_{j-1}}{2} + \frac{p_{j+1}}{2}
\]

\[
p_j = \frac{j}{n}
\]
Drunkard’s walk: probability

Pr[Tipsy goes home | he started at position $j$] = $\frac{j}{n}$

Pr[Tipsy falls into the river | he started at position $j$] = $\frac{n-j}{n}$
Drunkard’s walk: expected time

$s_j = \text{expected number of steps to finish the walk, starting at position } j$

$s_0 = 0$

$s_n = 0$

for $j \in [1, n - 1]$: $s_j = 1 + \frac{s_{j-1}}{2} + \frac{s_{j+1}}{2}$

$s_j = j(n - j)$
Markov Chains

- A (discrete time) **stochastic process** is a (finite or countably infinite) collection of random variables $X_0, X_1, X_2, \ldots$

- A discrete time stochastic process is a **Markov chain** if $\forall t \geq 1$ and $\forall$ values $a_0, a_1, \ldots, a_t$,
  
  $$
  \Pr[X_t = a_t | X_{t-1} = a_{t-1}, X_{t-2} = a_{t-2}, \ldots, X_0 = a_0] = \Pr[X_t = a_t | X_{t-1} = a_{t-1}]
  $$

  $$
  = P_{a_{t-1}, a_t}
  $$

  **Markov property or memoryless property**

  **Time-homogeneous property**

  *represent evolution of some random process over time*
Terminology

state space
the set of values the RVs can take, e.g. 0, 1, 2, ...

states visited by the chain
$X_0, X_1, ...$

transition probability from $a_{t-1}$ to $a_t$
$P_{a_{t-1}, a_t}$

Memoryless property:

• $X_t$ depends on $X_{t-1}$, but not on how the process arrived at state $X_{t-1}$.
• It does not imply that $X_t$ is independent of $X_0, ..., X_{t-2}$
  (only that this dependency is captured by $X_{t-1}$)
Set of vertices = state space
Directed edge \((i, j)\) iff \(P_{i,j} > 0\); the edge weight is \(P_{i,j}\)
Entry $P_{i,j}$ in matrix $P$ is the transition probability from $i$ to $j$.

For all rows $i$, the sum $\sum_{j \geq 0} P_{i,j} = 1$.

$P = \begin{pmatrix}
0 & \frac{1}{4} & 0 & \frac{3}{4} \\
\frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\
0 & 0 & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}$
Distribution of states

• Let $p_i(t)$ be the probability that the process is at state $j$ at time $t$. By Law of Total Probability,

$$p_j(t) = \sum_{i \geq 0} p_i(t - 1) \cdot P_{i,j}$$

• Let $\overline{p}(t) = (p_0(t), p_1(t), ...) \text{ be the (row) vector giving the distribution of the chain at time } t$.

$$\overline{p}(t) = \overline{p}(t - 1) \cdot P$$

• For all $m \geq 0$, we define the $m$-step transition probability

$$P_{i,j}^m = \Pr[X_{t+m} = j \mid X_t = i]$$

• Conditioning on the first transition from $i$, by Law of Total Probability,

$$P_{i,j}^m = \sum_{k \geq 0} P_{i,k} P_{k,j}^{m-1}$$
Distribution of states at time \( m \)

\[
P_{i,j}^m = \sum_{k \geq 0} P_{i,k} P_{k,j}^{m-1}
\]

- Let \( P^{(m)} \) be the matrix whose entries \((i, j)\) are the \( m \)-step transitional probabilities \( P_{i,j}^m \).
  \[
  P^{(m)} = P \cdot P^{(m-1)}
  \]
  By induction on \( m \),
  \[
  P^{(m)} = P^m
  \]
- For all \( t \geq 0 \) and \( m \geq 1 \),
  \[
  \tilde{p}(t + m) = \tilde{p}(t)P^m
  \]
What is the probability of going from state 0 to state 3 in exactly three steps?

\[
P = \begin{pmatrix}
0 & \frac{1}{4} & 0 & \frac{3}{4} \\
\frac{1}{2} & 0 & \frac{1}{3} & \frac{1}{6} \\
0 & 0 & 1 & 0 \\
0 & \frac{1}{2} & \frac{1}{4} & \frac{1}{4}
\end{pmatrix}
\]
What is the probability of going from state 0 to state 3 in exactly three steps?
Example

- We calculate the probability of the four events:

  - $0 - 1 - 0 - 3 \mid \text{Pr} = 3/32$
  - $0 - 1 - 3 - 3 \mid \text{Pr} = 1/96$
  - $0 - 3 - 1 - 3 \mid \text{Pr} = 1/16$
  - $0 - 3 - 3 - 3 \mid \text{Pr} = 3/64$

- Since they are mutually exclusive, the total probability is

  \[
  \text{Pr} = \frac{3}{32} + \frac{1}{96} + \frac{1}{16} + \frac{3}{64} = \frac{41}{192}
  \]
Example

Alternatively, we can calculate $P^3$

\[
\begin{pmatrix}
3/16 & 7/48 & 29/64 & 41/192 \\
5/48 & 5/24 & 79/144 & 5/36 \\
0 & 0 & 1 & 0 \\
1/16 & 13/96 & 107/192 & 47/192
\end{pmatrix}
\]

and find the entry $(0,3)$
Example 2

What is the probability of ending up in state 3 after three steps if we start in a uniformly random state?

Solution:

- Calculate

\[
\begin{pmatrix}
\frac{1}{4} & \frac{1}{4} & \frac{1}{4} & \frac{1}{4} \\
\frac{3}{4} & \frac{1}{2} & \frac{1}{6} & \frac{1}{4}
\end{pmatrix} P^3 = \begin{pmatrix}
\frac{17}{192} & \frac{47}{384} & \frac{737}{1152} & \frac{43}{288}
\end{pmatrix}
\]
Recall: A 2CNF formula is an AND of clauses

- Each clause is an OR of literals.
- Each literal is a Boolean variable or its negation.
- E.g. \((x_1 \lor x_2) \land (x_2 \lor x_3) \land (x_3 \lor x_4) \land (x_1 \lor x_4) \land (x_2 \lor x_4)\)

2SAT Problem (search version): Given a 2CNF formula, find a satisfying assignment if it is satisfiable.
Randomized Algorithm for 2SAT

Input: a 2CNF formula $\phi$ on $n$ variables

1. Start with an arbitrary truth assignment, e.g., all 0’s.
2. Repeat R times, terminating if $\phi$ is satisfied:
   a) Choose an arbitrary clause $C$ that is not satisfied.
   b) Pick a uniformly random literal in $C$ and flip its assignment.
3. If a satisfying assignment is found, return it.
4. Otherwise, return “unsatisfiable”.

Example: $\phi = (x_1 \lor \overline{x_2}) \land (x_2 \lor \overline{x_3}) \land (x_3 \lor \overline{x_4}) \land (x_1 \lor x_4) \land (\overline{x_2} \lor \overline{x_4})$

- Initial assignment: $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 0$
- Unsatisfied clause: $C = (x_1 \lor x_4)$
- Pick $x_1$ or $x_4$ and flip its value: $x_1 = 0, x_2 = 0, x_3 = 0, x_4 = 1$
- New unsatisfied clause: $C = (x_3 \lor \overline{x_4})$
- Pick $x_3$ or $\overline{x_4}$ and flip its value: $x_1 = 0, x_2 = 0, x_3 = 1, x_4 = 1$
• Only if $\phi$ is satisfiable, but we did not find a satisfying assignment in $R$ iterations (steps).

• We will analyze the number of steps necessary.

• Each step can be implemented to run in $O(n^2)$ time, since there are $O(n^2)$ clauses.
Analysis of the number of steps

- Let $S$ = a satisfying assignment of $\phi$.
- $A_i$ = an assignment to $\phi$ after $i$ steps
- $X_i$ = number of variables that have the same value in $A_i$ and $S$

When $X_i = n$, the algorithm terminates with a satisfying assignment.
(It could do it before $X_i = n$ if it finds another satisfying assignment.)

- If $X_i = 0$ then $X_{i+1} = 1$
  \[ \Pr[X_{i+1} = 1 \mid X_i = 0] = 1 \]

- If $X_i \in [1, n - 1]$ then $A_i$ disagrees with $S$ on 1 or 2 literals of $C$
  \[ \Pr[X_{i+1} = j + 1 \mid X_i = j] \geq 1/2 \]
  \[ \Pr[X_{i+1} = j - 1 \mid X_i = j] \leq 1/2 \]

$X_0, X_1, X_2, \ldots$ is not necessarily a Markov chain, since the probability of $X_{i+1} > X_i$ depends on whether $A_i$ and $S$ disagree on 1 or 2 literals of $C$ (which could depend on previous choices, not just $X_i$)
Creating a true Markov chain

- Define a Markov Chain $Y_0, Y_1, Y_2, \ldots$
  
  $Y_0 = X_0$

  \[
  \Pr[Y_{i+1} = 1 \mid Y_i = 0] = 1 \\
  \Pr[Y_{i+1} = j + 1 \mid Y_i = j] = 1/2 \\
  \Pr[Y_{i+1} = j - 1 \mid Y_i = j] = 1/2
  \]

- “Pessimistic version” of stochastic process $X_0, X_1, X_2, \ldots$

The expected time to reach $n$ is larger for $Y_0, Y_1, Y_2, \ldots$ than for $X_0, X_1, X_2, \ldots$
Expected time to reach $n$

$s_j = \text{expected number of steps to reach position } n, \text{ starting at position } j$

$s_0 = s_1 + 1$

$s_n = 0$

for $j \in [1, n - 1]$: $s_j = 1 + \frac{s_{j-1}}{2} + \frac{s_{j+1}}{2}$

$s_j = n^2 - j^2$

$s_0 = n^2$
2SAT algorithm: correctness

Theorem
If number of steps $R = 2an^2$ and $\phi$ is satisfiable, then the algorithm returns a satisfying assignment with probability at least $1 - 2^{-a}$.

Proof:

- The expected number of steps until ALG finds a satisfying assignment is $\leq n^2$, regardless of starting position.
- Brake $R$ into $a$ segments of $2n^2$
- Let $Z = \#$ steps ALG takes in segment $k$ without completion.