Randomness in Computing

LECTURE 26

CS 537

Last time

- Random walks on graphs
- Algorithm for s-t-PATH

Today

- Random walks on graphs: cover time (Matthews' Theorem)
- Parrondo's Paradox

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Reminders: Random walks

Given a connected, undirected graph $G = (V, E)$, define the following Markov chain

- states = vertices of the graph
- from each state v, the chain moves to a uniformly random neighbor of v

$$P_{uv} = \begin{cases} \frac{1}{d(u)} & \text{if } (u, v) \in E \\ 0 & \text{otherwise} \end{cases}$$

Hitting time, commute time, cover time

- The hitting time from u to v, denoted $h_{u,v}$, is the expected time to reach state v from state u.

- The commute time between u and v is $h_{u,v} + h_{v,u}$.

- The cover time of a graph $G = (V, E)$ is the maximum over $v \in V$ of the expected time for a random walk starting at v to visit all nodes in V.

Commute Time Lemma

If $(u, v) \in E$, the commute time $h_{u,v} + h_{v,u}$ is at most $2|E|$.

Cover Time Lemma

The cover time of G with n nodes and m edges is at most $2m(n-1)$.

Better bound on cover time

Cover Time Theorem (Matthew's Theorem)

The cover time C_G of a graph $G = (V, E)$ with n nodes is

$$C_G \leq H(n-1) \cdot \max_{u,v \in V : u \neq v} h_{u,v} .$$

Proof: Consider a random walk starting from a vertex u.

- Let $Z_1, \dots, Z_n$ be a uniformly random permutation of V.

Idea 1: Use phases, like in Coupon Collector

- Let T_j for $j \in [n]$ be the first time when all vertices in $Z_1, \dots, Z_j$ have been visited (``collected'')
- Let L_j for $j \in [n]$ be the last vertex from $\{Z_1, \dots, Z_j\}$ to have been collected

By Principle of Deferred Decisions

Idea 2: Since $Z_1, \dots, Z_n$ is a uniformly random permutation, L_j is equally likely to be any of the vertices in $\{Z_1, \dots, Z_j\}$

Parrondo's Paradox

Two losing games can be combined to make a winning game

- For a coin c, let p_c be the probability of HEADS,
  $q_c = 1 - p_c$ be the probability of TAILS
- We consider repeated games.

One round of **Game A** with coin a
Toss the coin • win \$1 if HEADS • lose \$1 if TAILS

- Example: if $p_a = 0.49$, the expected loss is

CS 537 Game B

Two losing games can be combined to make a winning game

- For a coin c, let p_c be the probability of HEADS,
  $q_c = 1 - p_c$ be the probability of TAILS
- We repeatedly flip two coins

One round of **Game B** with coins b and c

Let W = # of wins in previous rounds;
L = of losses in previous rounds
If $W - L$ is divisible by 3, flip coin b; else flip coin c.
- win \$1 if HEADS
- lose \$1 if TAILS

$W - L =$ ***total winnings so far (can be negative)***

- Example: $p_b = 0.09, p_c = 0.74$

If you flip coin b one third of the time

But you don't!

$$p_{win} = \frac{1}{3} \cdot \frac{9}{100} + \frac{2}{3} \cdot \frac{74}{100} = \frac{157}{300} > \frac{1}{2}$$

Analysis of Game B

- For a coin c, let p_c be the probability of HEADS,
  $q_c = 1 - p_c$ be the probability of TAILS

Idea: Combine rounds into phases, each lasting until we win or lose \$3

Idea 2: To analyze a phase, pair up winning and losing sequences

- Consider any sequence of states that starts at 0 and ends at 3 before reaching -3.
- $f(s)$ is created by negating every number starting from the last 0
- Then $f(s)$ is 1-to1.

Lemma (for Game B)

For any sequence s of states that starts at 0 and ends at 3 before reaching -3:

$$\frac{\Pr[\,s \quad \text{occurs}]}{\Pr[f(s)\ \text{occurs}]} = \frac{p_b \cdot p_c^2}{q_b \cdot q_c^2}.$$

Analysis of Game B

Lemma (for Game B)

For any sequence s of moves that starts at 0 and ends at 3 before reaching -3:

$$\frac{\Pr[\,s \quad \text{occurs}]}{\Pr[f(s)\text{ occurs}]} = \frac{p_b \cdot p_c^2}{q_b \cdot q_c^2}.$$

- Let S be the set of all sequences of states that start at 0 and end at 3 before reaching -3.

$$\frac{\Pr[3\text{ is reached before } -3]}{\Pr[-3\text{ is reached before }3]} = \frac{\sum_{s\in S}\Pr[\,s \quad \text{occurs}]}{\sum_{s\in S}\Pr[f(s)\text{ occurs}]}$$

- Example: $p_b = 0.09, p_c = 0.74$

We get $\frac{12{,}321}{15{,}375} < 1$: more likely to lose than win

Combining the two games

Two losing games can be combined to make a winning game

One round of Game C with coins a, b, c

Flip a fair coin
- Play Game A if HEADS
- Play Game B if TAILS

- Example: if $p_a = 0.49$, the expected loss is