Randomness in Computing

Lecture 27

CS 537

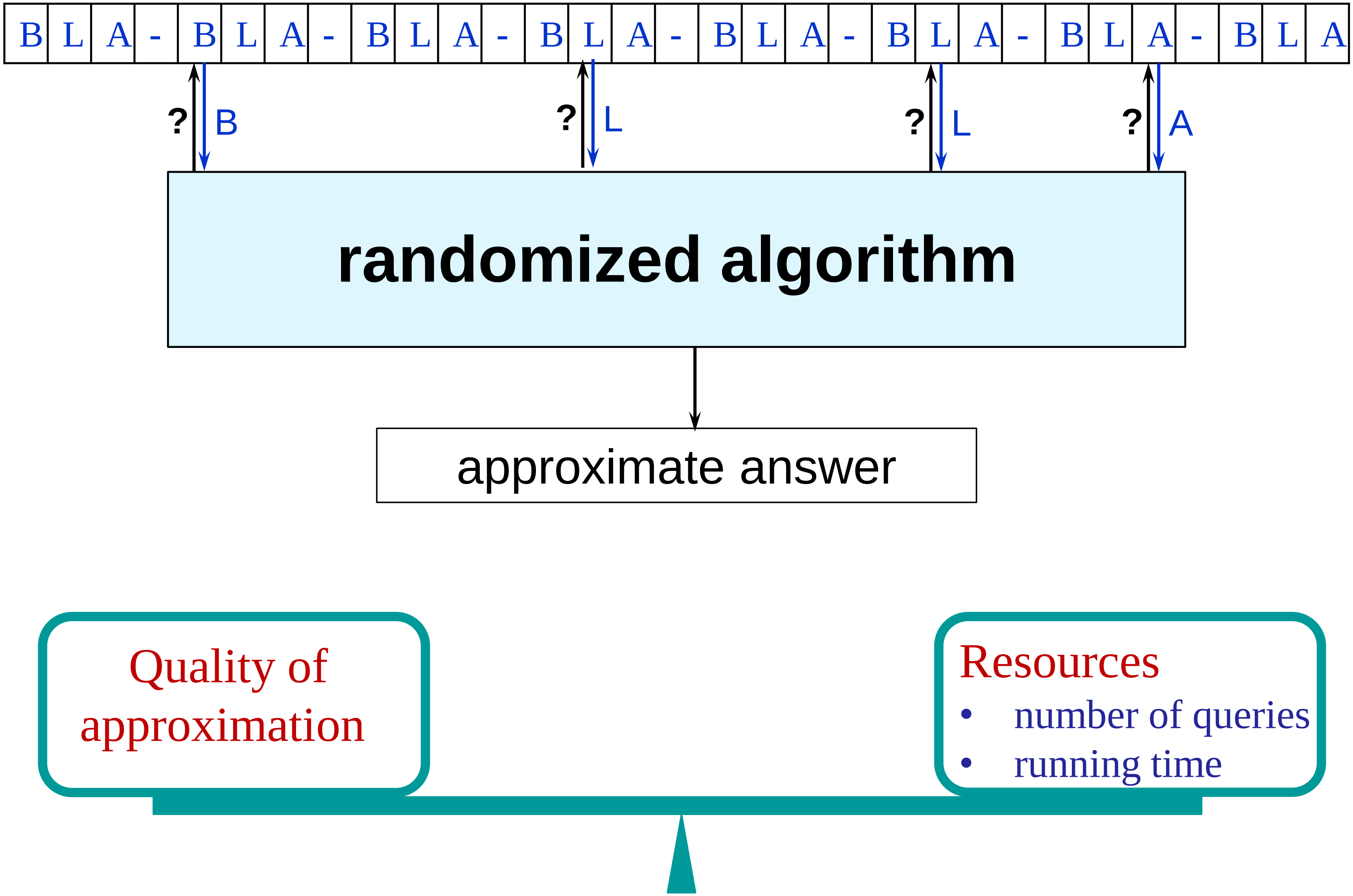
Last time

- Stationary distributions
- Random walks on graphs
- Algorithm for s-t-PATH

Today

- Sublinear algorithms
- Differential privacy

A Sublinear-Time Algorithm



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Goal: Fundamental Understanding of Sublinear Computation

- What computational tasks?
- How to measure quality of approximation?
- What type of access to the input?
- Can we make our computations robust (e.g., to noise or erased data)?

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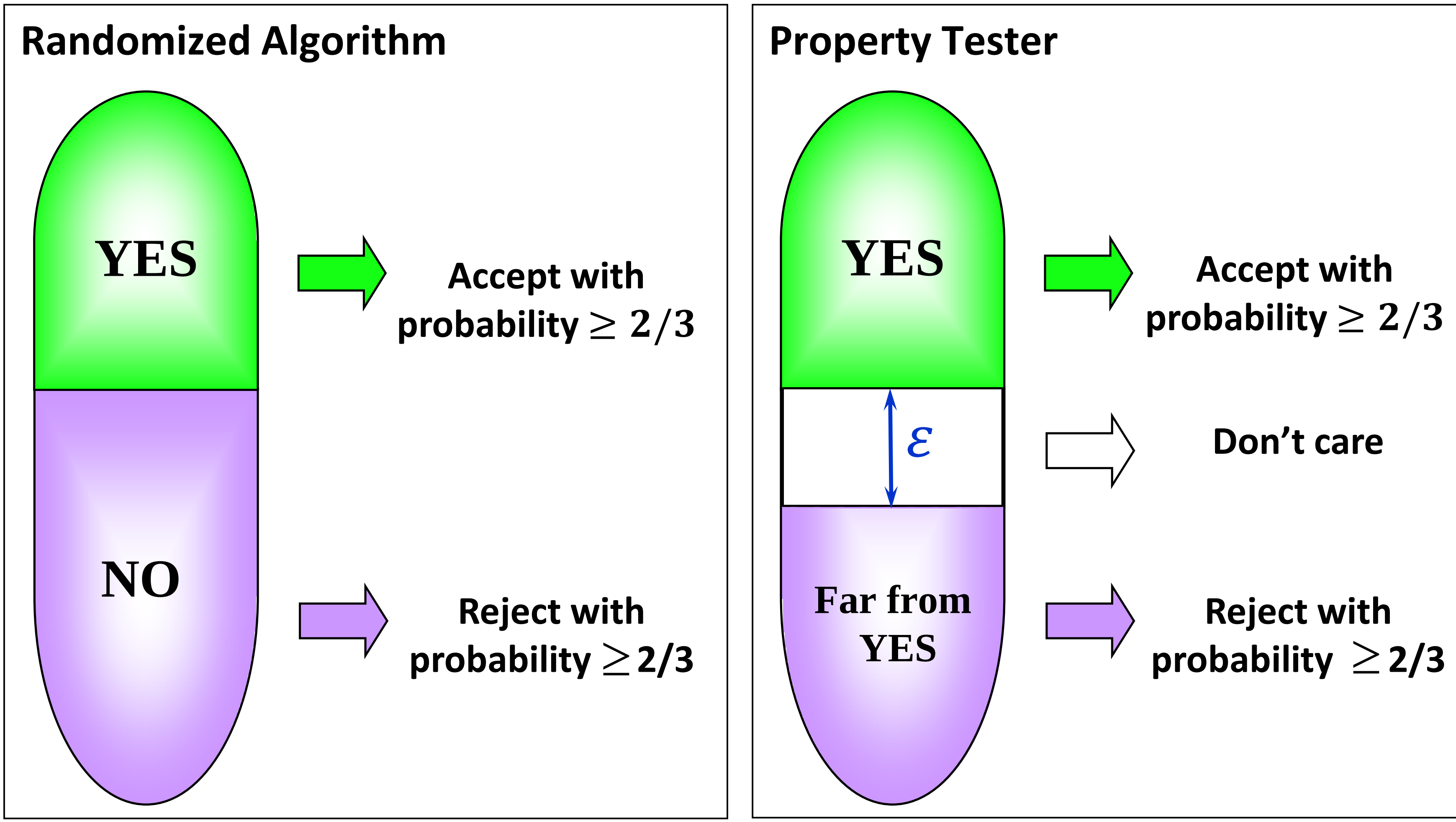
Fundamental Computational Tasks

- Property testing
  - need to answer YES or NO
  - ➢ **intuition: only require correct answers on two sets of instances that are very different from each other**
- Learning
  - need an approximate representation of an object
  - ➢ **input is from a given class (or is close to it)**
- Classical approximation
  - need to compute a value
  - ➢ **output should be close to the desired value**

Property Testing: Definition

[Rubinfeld Sudan, Goldreich Goldwasser Ron]



ε-far = differs in many places ($\geq \varepsilon$ fraction of places)

Example: Lipschitz Testing [Jha R]

Input: a list of n numbers $x_1, x_2, \dots, x_n$

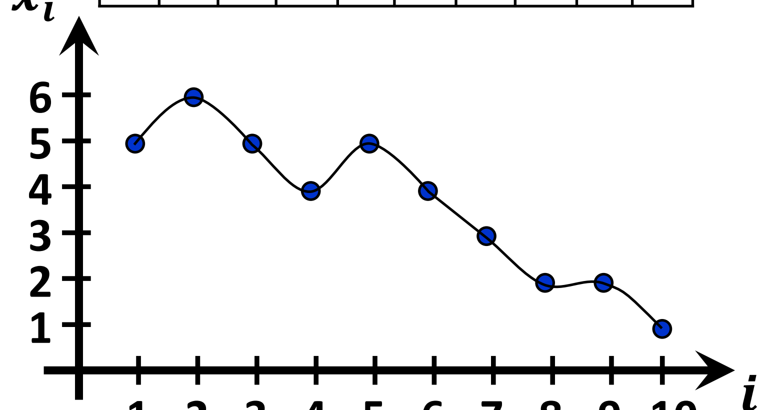
- A list of numbers is Lipschitz if $|x_{i+1} - x_i| \le 1$ for all i.
- Question: Is the list Lipschitz?

  Requires reading entire list: $\Omega(n)$ time
- Approximate version: Is the list Lipschitz or ε-far from Lipschitz?

  (An ε fraction of x_i's have to be changed to make it Lipschitz.)

  Our result: $O((\log n)/\varepsilon)$ time

5	6	5	4	5	4	3	2	2	1



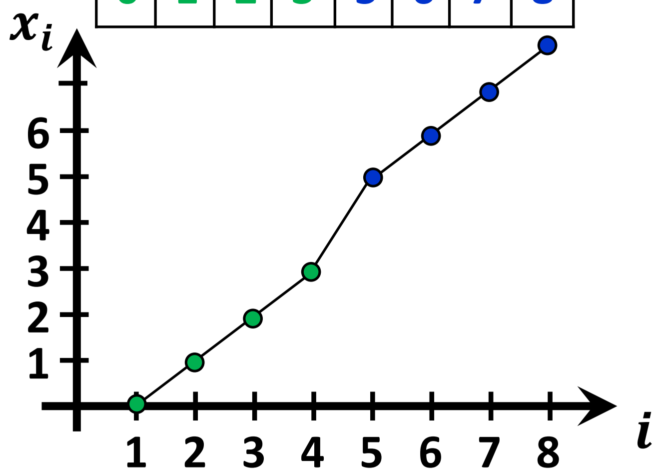
Lipschitz Testing: Attempts

1. **Test**: Pick a random i and reject if $|x_{i+1} - x_i| > 1$

Fails on:

0	1	2	3	5	6	7	8

← 1/2-far from Lipschitz

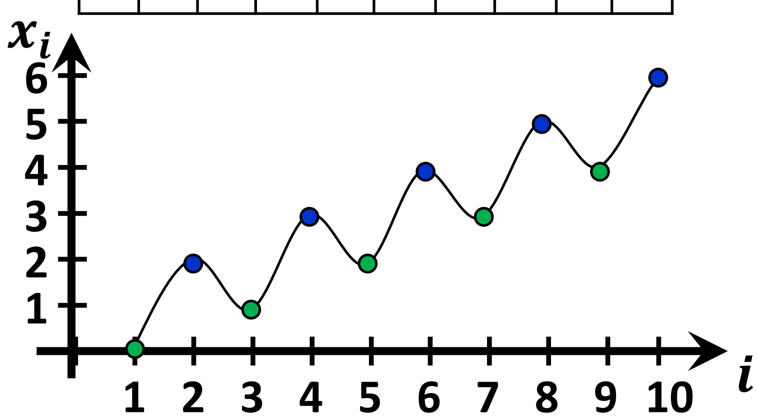


2. **Test**: Pick random $i < j$ and reject if $|x_j - x_i| > j - i$

Fails on:

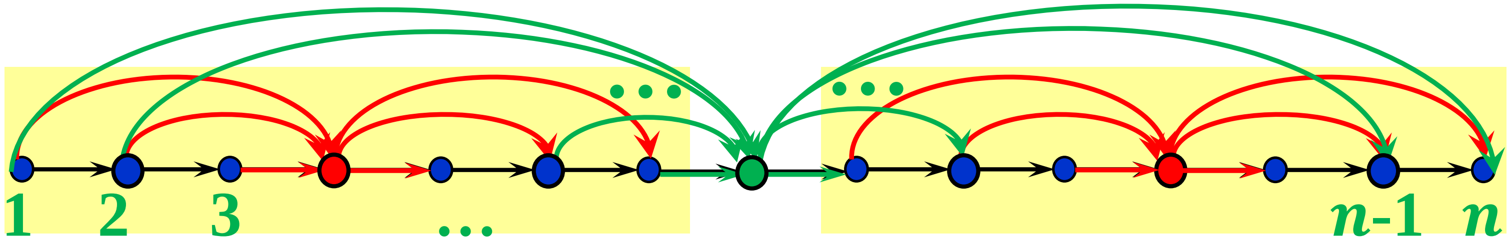
0	2	1	3	2	4	3	5	4	6

← 1/2-far from Lipschitz



Is a list Lipschitz or ε-far from Lipschitz?

Idea: Associate positions in the list with vertices of the directed line.



Construct a graph (2-spanner)

$\leq n \log n$ edges

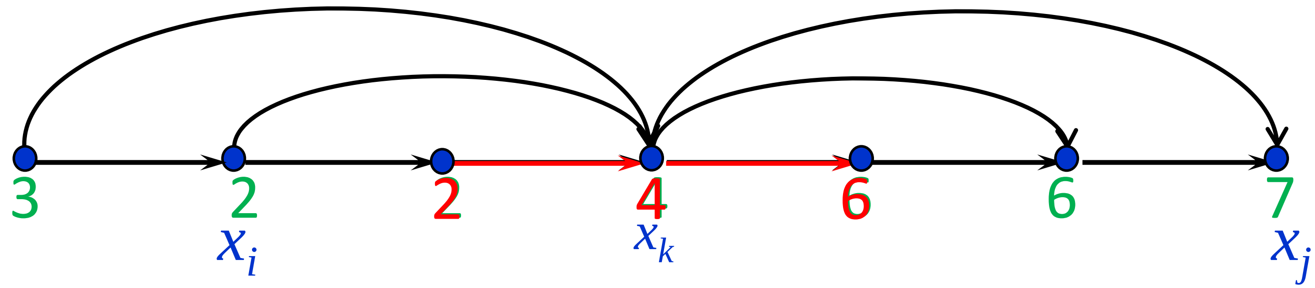
[Bhattacharyya Grigorescu Jung **R** Woodruff]

- by adding a few "shortcut" edges (i, j) for $i < j$
- where each pair of vertices is connected by a path of length at most 2

Is a list Lipschitz or ε-far from Lipschitz?

Test

Pick a random edge (i, j) from the 2-spanner and **reject** if $|x_j - x_i| > j - i$.



Analysis:

- Call a pair (i, j) **violated** if $|x_j - x_i| > j - i$, and **satisfied** otherwise.
- If i is an endpoint of a **violated** edge, call x_i **bad**. Otherwise, call it **good**.

Claim 1. All pairs of **good** numbers are satisfied.

Proof: Consider any two good numbers, x_i and x_j.

They are connected by a path of (at most) two **satisfied** edges $(i, k), (k, j)$

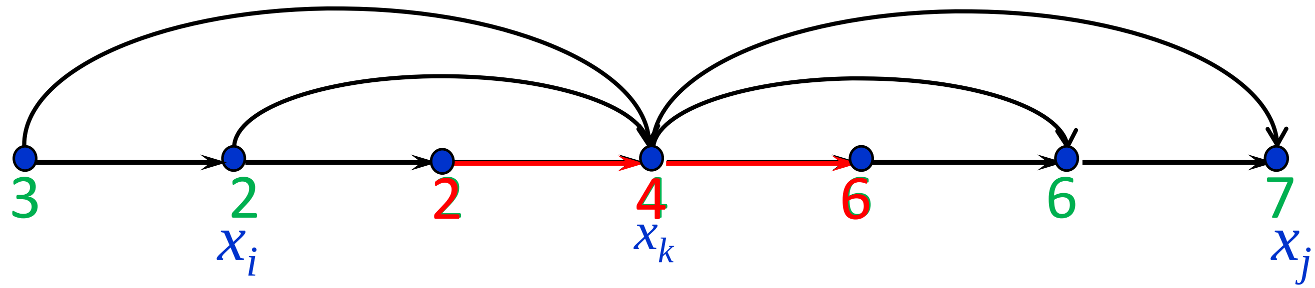
$\Rightarrow |x_k - x_i| \le k - i$ and $|x_j - x_k| \le j - k$

$\Rightarrow |x_j - x_i| \le |x_j - x_k| + |x_k - x_i| \le (j - k) + (k - i) = j - i$

Is a list Lipschitz or ε-far from Lipschitz?

Test

Pick a random edge (i, j) from the 2-spanner and **reject** if $|x_j - x_i| > j - i$.



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Claim 1. All pairs of **good** numbers are satisfied.

Claim 2. An ε-far list **violates** $\geq \varepsilon/(2 \log n)$ fraction of edges in 2-spanner.

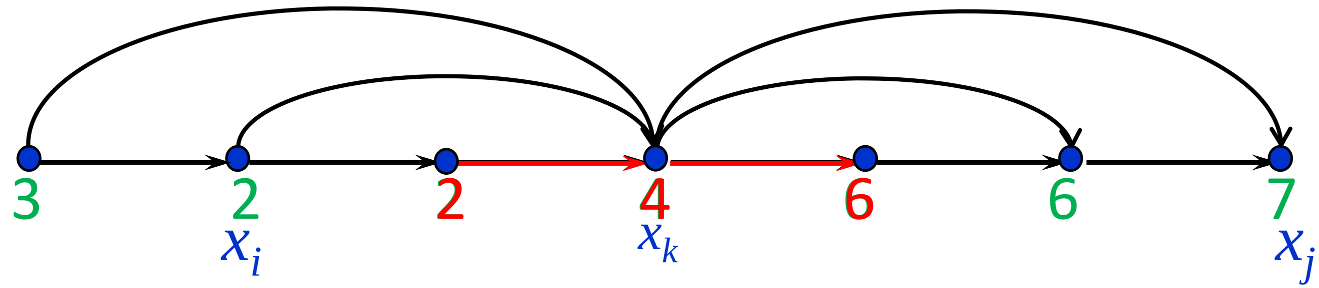
Proof: If a list is ε-far from Lipschitz, it has $\geq \varepsilon n$ **bad** numbers. (Claim 1)

- Each **violated** edge contributes 2 **bad** numbers.
- 2-spanner has $\geq \frac{\varepsilon n}{2}$ **violated** edges out of $n \log n$.

Is a list Lipschitz or ε-far from Lipschitz?

Test

Pick a random edge (i, j) from the 2-spanner and **reject** if $|x_j - x_i| > j - i$.



Analysis:

- Call a pair (i, j) **violated** if $|x_j - x_i| > j - i$, and **satisfied** otherwise.

Claim 2. An ε-far list **violates** $\geq \varepsilon/(2 \log n)$ fraction of edges in 2-spanner.

Algorithm

Sample $\frac{4 \log n}{\varepsilon}$ edges (x_i, x_j) from the 2-spanner and **reject** if $|x_j - x_i| > j - i$.

Guarantee: All Lipschitz lists are accepted. ✓

All lists that are ε-far from Lipschitz are rejected with probability $\geq 2/3$. ✓

Time: O((log n)/^{2}) ✓

Testing if a List is Lipschitz: Summary

- [Jha R]:

  We can determine if a list of n numbers is

  Lipschitz or ε-far from Lipschitz

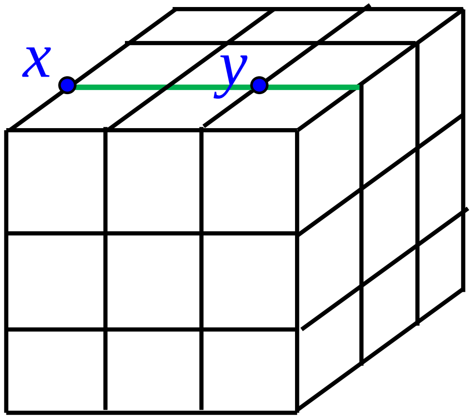
  in $O\left(\frac{\log n}{\varepsilon}\right)$ time.

- [Jha R, Blais R Yaroslavtsev, Chakrabarty Dixit Jha Seshadhri]:

  This cannot be improved.

Testing Properties of High-Dimensional Functions

In polylogarithmic time, we can test a large class of properties of functions $f\colon\{1,\ldots,n\}^d \to \mathbb{R}$, including:



- Lipschitz property [Jha **R**]
- Monotonicity [Goldreich Goldwasser Lehman Ron, Dodis Goldreich Lehman **R** Ron Samorodnitsky]
- Bounded-derivative properties [Chakrabarty Dixit Jha Seshadhri]
- Unateness [Baleshzar Chakrabarty Pallavoor **R** Seshadhri]

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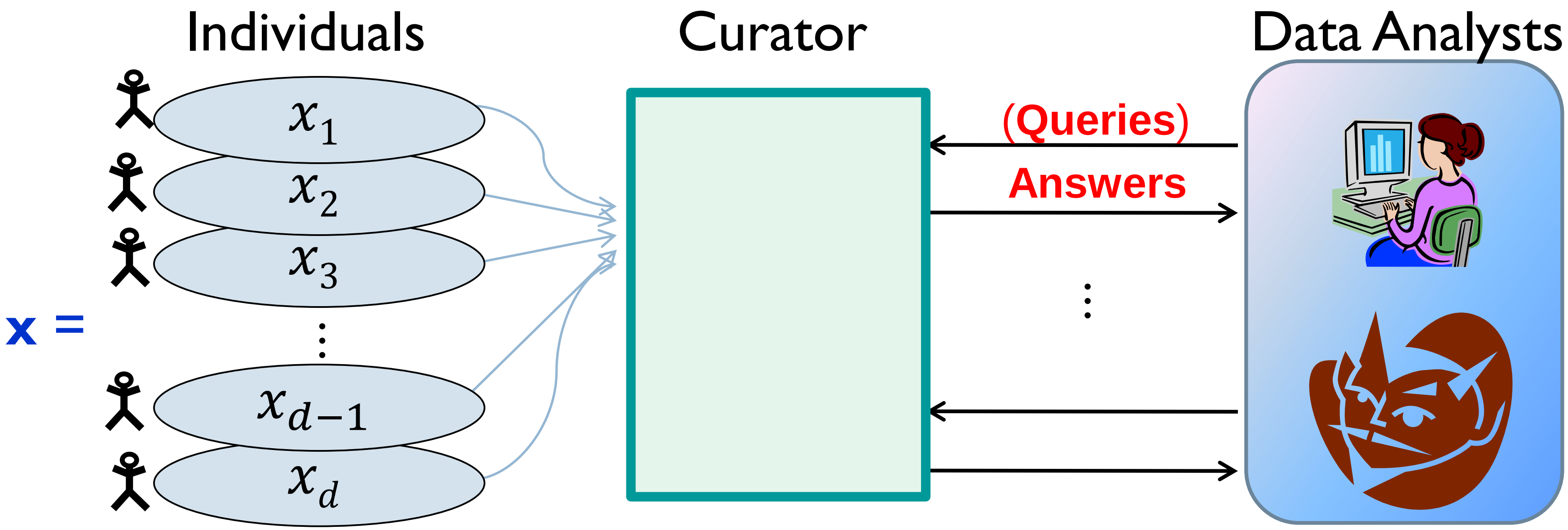
Sublinear Algorithms: Summary

- Many problems admit sublinear-time algorithms
- Algorithms are often simple
- Analysis requires creation of interesting combinatorial, geometric and algebraic tools
- Unexpected connections to other areas
- Many open questions

Private Data Analysis



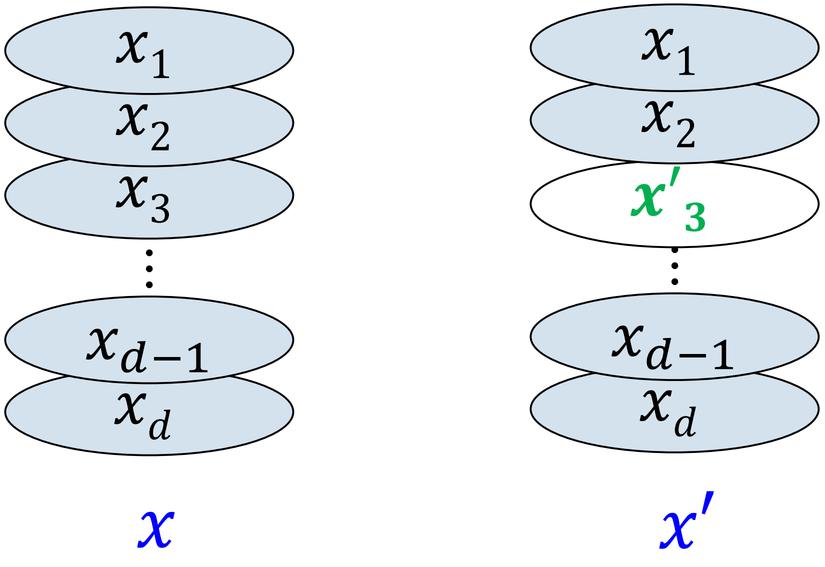
Typical examples: census, medical studies, what big companies want to publish about our data…

Two conflicting goals

- Protect privacy of individuals
  - **Differential privacy** [Dwork McSherry Nissim Smith 06]
- Give accurate answers

Neighboring Datasets

Two datasets x, x' are **neighbors** if they differ in one person's data.



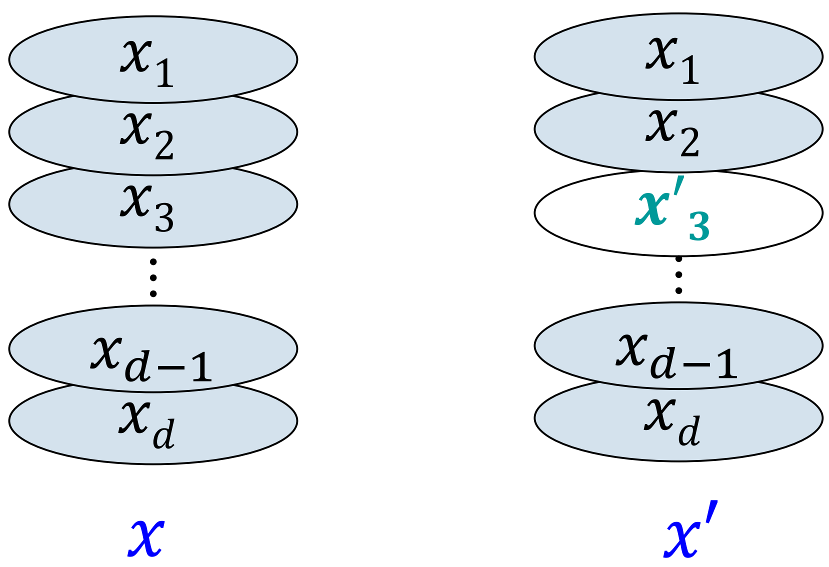
Differential Privacy [Dwork McSherry Nissim Smith]

Privacy Definition

An algorithm A is ϵ-differentially private if

for all pairs of **neighbors** $\boldsymbol{x}, \boldsymbol{x}'$ and all sets of answers **S**:

$$\mathbf{Pr}[\boldsymbol{A}(\boldsymbol{x}) \in \boldsymbol{S}] \leq \boldsymbol{e}^{\boldsymbol{\epsilon}}\, \mathbf{Pr}[\boldsymbol{A}(\boldsymbol{x}') \in \boldsymbol{S}]$$



Properties of Differential Privacy

- Composition:

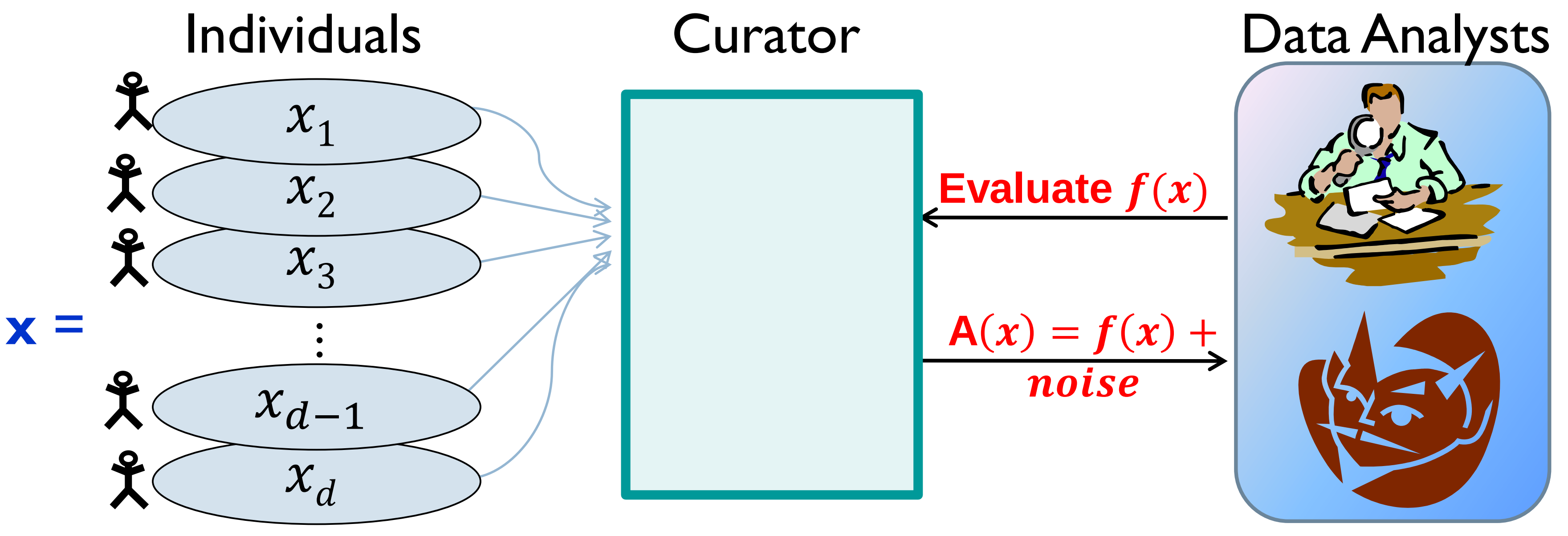
If algorithms A_1 and A_2 are ϵ-differentially private **then** algorithm that outputs $(A_1(x), A_2(x))$ is 2ϵ-differentially private

- Meaningful in the presence of arbitrary external information

Output Perturbation

Frameworks for designing differentially private algorithms

Output Perturbation



CS 537 Global Sensitivity Framework

Global sensitivity of a function f is

$$\boldsymbol{GS_f} = \max_{\textbf{neighbors}\, x,x'} |f(x) - f(x')|.$$

Example: $x_1, \ldots, x_n \in [0,1]$, $\text{ave}(x) = \frac{x_1 + \cdots + x_n}{n}$

- $GS_{\text{ave}} = ?$

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- $GS_{\text{ave}} = 1/n$

Theorem [Dwork McSherry Nissim Smith]

If $A(x) = f(x) + Lap\left(\frac{GS_f}{\epsilon}\right)$ then A is ϵ-differentially private.

Global Sensitivity: Noise Distribution

Laplace Mechanism Theorem [Dwork McSherry Nissim Smith]

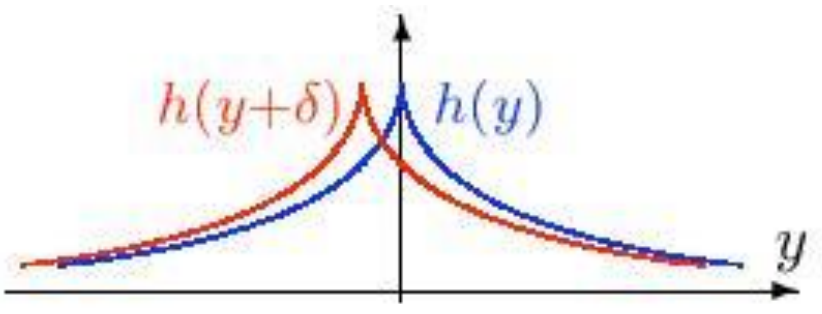
If $A(x) = f(x) + Lap\left(\frac{GS_f}{\epsilon}\right)$ then A is ϵ-differentially private.

Laplace distribution Lap(λ) has density $h(y) = \frac{1}{2\lambda} \cdot e^{-\frac{|y|}{\lambda}}$

(mean 0, standard deviation $\sqrt{2} \cdot \lambda$)

Sliding Property of $Lap\left(\frac{GS_f}{\epsilon}\right)$

for all y, δ: $\frac{h(y)}{h(y+\delta)} \le e^{\epsilon \cdot \frac{|\delta|}{GS_f}}$



When is Laplace Mechanism Useful?

- Laplace mechanism is always private.
- When is it accurate?

Example: $x_1, \dots, x_n \in [0,1],\ \text{ave}(x) = \frac{x_1 + \cdots + x_n}{n}$

- $GS_{\text{ave}} = 1/n$ Noise= $\text{Lap}\left(\frac{1}{\epsilon n}\right)$

Accurate when GS is low

(and n, the size of the database, is sufficiently large)

Can Global Sensitivity Be Too High?

Example: $x_1, \ldots, x_n \in [0,1]$, $median(x)$ is median of $x_1, \ldots, x_n$.

- GS_{median} = ?

$$x = \underbrace{0 \ldots 0}_{\frac{n-1}{2}} \; 0 \; \underbrace{1 \ldots 1}_{\frac{n-1}{2}} \qquad x' = \underbrace{0 \ldots 0}_{\frac{n-1}{2}} \; 1 \; \underbrace{1 \ldots 1}_{\frac{n-1}{2}}$$

$$median(x) = 0 \qquad median(x') = 1$$

- Noise: Lap$\left(\frac{1}{\epsilon}\right)$ *Too much noise!*
- But for most neighboring datasets x and x',

  $median(x) - median(x')$ is small
- Can we add less noise on ``good'' datasets?

Smooth Sensitivity Framework

[Nissim Raskhodnikova Smith]

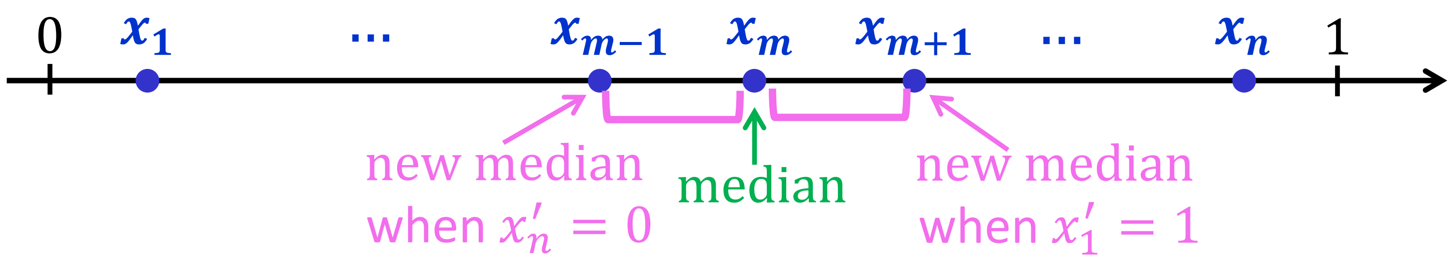
Local Sensitivity

Local sensitivity of a function f at point x is

$$LS_f(x) = \max_{x' : \textbf{neighbor } of\ x} |f(x) - f(x')|.$$

Relationship to GS: $GS_f = \max_{\textbf{datasets } x} LS_f(x)$

Example: $median\ of\ 0 \leq x_1 \leq \cdots \leq x_n \leq 1$ for odd n.



- $\text{LS}_{\text{median}}(x)$= ?

Local Sensitivity

Local sensitivity of a function f at point x is

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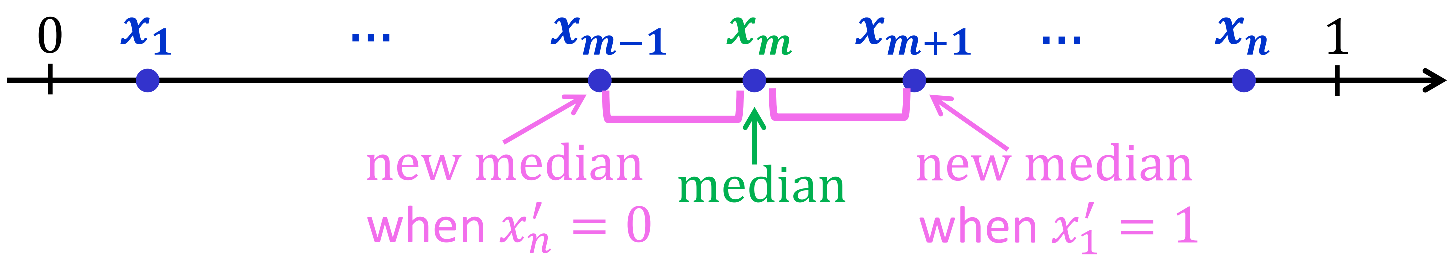


- $\mathrm{LS}_{\text{median}}\ (x)$= $\max(x_{m+1} - x_m,\ x_m - x_{m-1})$

Goal: Release $f(x)$ with less noise when $LS_f(x)$ is lower.

First Attempt: Local Sensitivity

Noise with magnitude proportional to $LS_f(x)$ instead of GS_f?

Problem: noise magnitude might reveal information.

Example: median

$$x = \underbrace{0 \ldots 0}_{\frac{n-3}{2}}\ 000\ \underbrace{1 \ldots 1}_{\frac{n-3}{2}} \qquad x' = \underbrace{0 \ldots 0}_{\frac{n-3}{2}}\ 001\ \underbrace{1 \ldots 1}_{\frac{n-3}{2}}$$

$$median(x) = 0 \qquad median(x') = 0$$

$$LS_{median}(x) = 0 \qquad LS_{median}(x') = 1$$

$$\Pr[A(x) = 0] = 1 \qquad \Pr[A(x') = 0] = 0$$

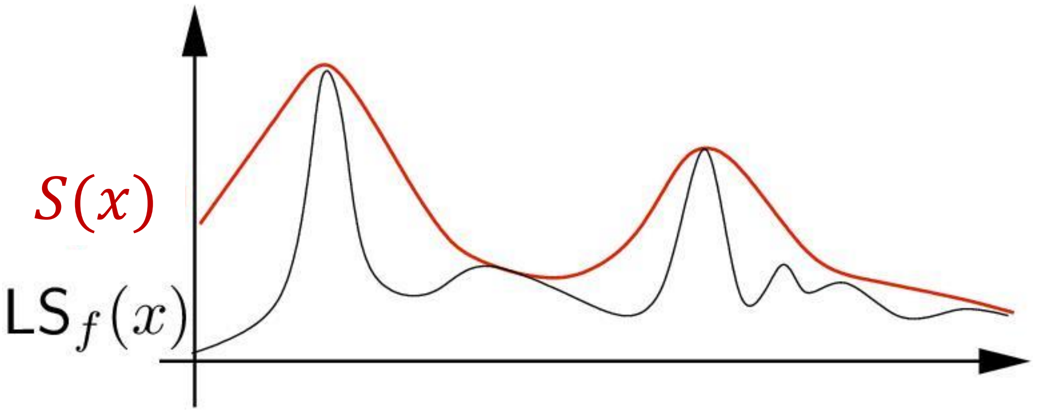
A is not differentially private

- Idea: make noise magnitude an ``insensitive'' function

Smooth Bounds on Local Sensitivity

Design sensitivity function $S(x)$

- $S(x)$ is an ϵ-smooth upper bound on $LS_f(x)$ if
  - for all x: $S(x) \geq LS_f(x)$
  - for all neighbors x, x': $S(x) \leq e^{\epsilon} S(x')$



Theorem

If $A(x) = f(x) + noise\left(\frac{S(x)}{\epsilon}\right)$ then A is (ϵ', δ)-diff. private.

Example: GS_f is a smooth bound on $LS_f(x)$.

Smooth Sensitivity

- For two datasets x and y, let $dist(x, y)$= $|\{i: x_i \neq y_i\}|$
- **Smooth sensitivity** $S_f^*(x) = \max_{\textbf{datasets } y} LS_f(y) \cdot e^{-\epsilon \cdot dist(x,y)}$.
- Intuition: $S_f^*(x)$ is low when x is far from sensitive datasets

Lemma

1. Smooth sensitivity is an ϵ-smooth upper bound on LS_f.
2. For every ϵ-smooth upper bound S on LS_f:

$$S_f^*(x) \leq S(x) \text{ for all } x.$$

Computing Smooth Sensitivity

Recall: **Smooth sensitivity** $S_f^*(x) = \max_{y} LS_f(y) \cdot e^{-\epsilon \cdot dist(x,y)}$.

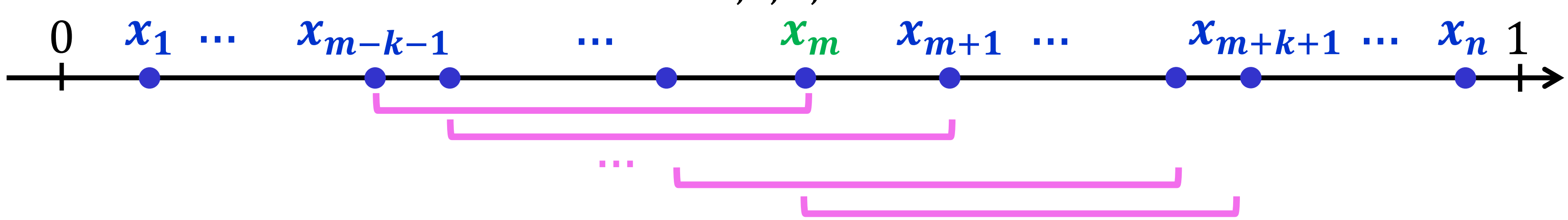
Observation

$$S_f^*(x) = \max_{k=0,1,\ldots,n} LS_f^k(x) \cdot e^{-\epsilon \cdot k},$$

where $LS_f^k(x) = \max_{y: dist(x,y) \le k} LS_f(y)$.

Example: median

$$LS_{\text{median}}^k(x) = \max_{t=0,1,\ldots,k+1} (x_{m+t+k+1} - x_{m+t})$$



This gives $O(n^2)$ time algorithm for computing $S_{\text{median}}^*(x)$.

(It can be computed in time $O(n \log n)$.)

Conclusion: Calibrating Noise

- Adding noise proportional to local sensitivity is not safe.
- Smooth sensitivity framework allows one to calibrate noise to the input dataset.
  - Requires understanding combinatorial structure of the problem.
- There are other frameworks based on local sensitivity:
  - Propose-Test-Release [Dwork Lei, Karwa R Smith Yaroslavtsev]
  - Sample-and-Aggregate [Nissim R Smith]

Conclusion: Differential Privacy

- a rigorous and widely applicable notion of privacy
- is defined in terms of algorithm
- requires the algorithm to be randomized
- puts a restriction on the algorithm, requiring that output distributions on neighboring datasets be close
- is used in 2020 Census, by Apple and Google