Homework 1 – Due Thursday, September 9, 2021 by midnight

Submit your solutions on Gradescope. Don’t forget to include information about your collaborators (or say “Collaborators: none”) for each problem.

**Page limit**  You can submit at most 2 pages per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first two pages will be graded.

**Exercises**  Please practice on exercises in Chapter 1 of Mitzenmacher-Upfal.

**Problems**

0. (0 points) The following steps are required to get you started in the course.

(a) Sign up on piazza at [piazza.com/bu/fall2021/cs537](https://piazza.com/bu/fall2021/cs537) using your BU email address.

(b) Sign up on Gradescope using your BU email address and the code **X37GYP**.

(c) Read and sign the Collaboration and Honesty Policy and submit it on Gradescope. We will be able to grade your homework only after you complete this step.

(d) (Nameplate) Please print out (or make by hand) a nameplate with your name and bring it to every lecture and discussion. A template is available at the bottom of the course web page.

(e) Check out the following links and resources:

   i. course webpage: [https://cs-people.bu.edu/sofya/cs537/](https://cs-people.bu.edu/sofya/cs537/)

   ii. supplementary textbook to review proof techniques:


(f) Familiarize yourself with the homework template files at the bottom of the course web page. Note that each problem has to be submitted separately and each must include a note about collaborators (even if you did the problem by yourself).

1. (Probability review, 10 points) For each part below, explain how you got your answer. Nearly all points will be allocated for your explanation. Professor Sofya shows a magic trick in class. She asks each student to pick a number between 1 and 10. Then she deals a shuffled standard deck of 52 cards face up, one at a time. For each student, their first key card is the one at the position they choose in advance. The value of this card determines how many cards are dealt out to the next key card. For example, if the key card is a 10, the student counts off ten cards, the last being the new key card. An ace counts as 1 and a royal card (a jack, a queen, or a king) counts as 5. The process is repeated until Professor Sofya announces her guess. (You can ask her to perform the trick if you haven’t seen it.)

(a) Assuming that two students pick their initial numbers uniformly at random, what is the probability that they pick the same number?

(b) Assuming that five students pick their initial numbers uniformly at random, what is the probability that they are all different?

(c) Suppose that Fabian initially picks 1, and Iden initially picks 2. What is the probability that Iden’s first key card is also a key card for Fabian?

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(d) Suppose that Fabian initially picks 1, and Iden initially picks 2. What is the probability that Iden’s second key card is also the second key card for Fabian?

(e) What’s the probability that the first five cards dealt in the deck form a full house (three of one rank and two of another rank)?

2. (Chess board, 10 points) Consider an $8 \times 8$ chess board in which the rows are numbered from 1 to 8, and likewise for the columns. And, as is usual for a chess board, the squares are alternately colored black and white. The squares of this chess board form the elements of a sample space in which all of the 64 squares on the chess board are equally likely; that is, all have probability $\frac{1}{64}$.

For each of the following pairs of events $A$ and $B$, determine if the two events are independent and prove your answer is correct:

(a) $A$ is the event that a white square is chosen and $B$ is the event that a black square is chosen.

(b) $A$ is the event that a square from an even numbered row is chosen and $B$ is the event that a square from an even numbered column is chosen.

(c) $A$ is the event that a white square is chosen and $B$ is the event that a square from an even numbered column is chosen.

Determine if the following three events $A$, $B$ and $C$ are mutually independent and prove your answer is correct:

(d) $A$ is the event that a square from an even numbered row is chosen, $B$ is the event that a square from an even numbered column is chosen, and $C$ is the event that a white square is chosen.

3. (Homework assignments, 10 points)

(a) You start working on the first homework as soon as it is assigned to you. Every time a new homework is assigned, you switch to working on it with a certain probability and keep working on your current homework with the remaining probability. Specifically, when homework $k$ is assigned, you switch to working on this homework with probability $\frac{1}{k}$. Prove that you are equally likely to work on any homework assigned so far. (In other words, the homework you are working on is uniformly distributed over all homework assignments so far.)

(b) Suppose that your friend has a similar strategy, except that when the $k$th homework is assigned, for $k \geq 2$, she switches to working on it with probability $1/2$. Describe the distribution of the homework assignment she is working on.

4* (Optional, no collaboration) Professor Sofya offers you a chance to get an A++ in the course by playing a simple game. She gives you 50 cookies with white chocolate chips, 50 cookies with dark chocolate chips, and 2 empty identical containers. She asks you to put these 100 cookies into the two containers. For each cookie, you must decide which container to put it in. Then you will be blindfolded, and the positions of the containers will be mixed up. After that, you can choose one container and take ONE cookie. (All cookies feel identical to the touch, and you are not allowed to modify them in any way.) If the cookie you took has dark chocolate chips, you get an A++ for the course. Otherwise, you have to do all homework assignments and take three exams to earn at most A.

Figure out which cookies to put in each container so that you have the greatest probability of choosing a cookie with dark chocolate chips.