Homework 1 – Due Thursday, September 5, 2024 by midnight

Submit your solutions on Gradescope. Don't forget to include information about your collaborators (or say "Collaborators: none") for each problem.

Page limit You can submit **at most** 2 pages per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first two pages will be graded.

Exercises Please practice on exercises in Chapter 1 of Mitzenmacher-Upfal.

Problems

0. (**Do ASAP, 1 point**) The following steps are required to get you started in the course.

   (a) Sign up on piazza at `piazza.com/bu/fall2024/cs537` using your BU email address.

   (b) Sign up on Gradescope using your BU email address and the code **7EX6ZE**.

   (c) Read and sign the Collaboration and Honesty Policy and submit it on Gradescope. We will be able to grade your homework only after you complete this step.

   (d) (**Nameplate**) Please print out (or make by hand) a nameplate with your name and bring it to every lecture and discussion. A template is available at the bottom of the course web page.

   (e) Check out the following links and resources:

      i. course webpage: `https://cs-people.bu.edu/sofya/cs537/`;

      ii. supplementary textbook to review proof techniques:
      Richard Hammack. *Book of Proof*: `http://www.people.vcu.edu/~rhammack/BookOfProof/`

   (f) Familiarize yourself with the homework template files at the bottom of the course webpage. Note that each homework assignment must include a note about collaborators (even if you did it by yourself). You will need to mark where each problem is in your file when you submit it on Gradescope.

1. (**Probability review, 10 points**) **For each part below, explain how you got your answer. Nearly all points will be allocated for your explanation.** Professor Sofya shows a magic trick in class. She asks a student volunteer to pick an integer between 1 and 10. Then she deals a shuffled standard deck of 52 cards face up, one at a time. The first focus card for the student volunteer is the one at the position they chose in advance. The value of this card determines how many cards are dealt out to the next focus card. For example, if the focus card is a 10, the student counts off ten cards, the last being the new focus card. An ace counts as 1 and a royal card (a jack, a queen, or a king) counts as 5. The process is repeated until Professor Sofya guesses the current focus card. (You can ask her to perform the trick if you haven't seen it[1].)

   (a) What is the probability that the first three cards dealt in the deck are all aces?

   (b) What is the probability that the first four cards dealt in the deck all count as 5? (Recall that 5s, jacks, queens, and kings count as 5.)

[1] This is the only magic card trick she knows that is based on probability. But there is a huge number of tricks based on math more generally.

(c) If the student volunteer chooses 1, what is the probability that the card in position 4 is one of their focus cards?

(d) Now the magic trick is performed with 11 student volunteers simultaneously. If they all pick their initial integers uniformly and independently at random, what is the probability that they all pick the same number?

(e) If all 11 volunteers pick their initial integers uniformly and independently at random, what is the probability that all 10 possibilities are selected (by somebody)?

2. (**Algorithms review, 10 points**) This problem is about the same magic trick at the previous problem. However, for this problem, please assume that each card is picked uniformly at random from the deck **with replacement**, that is, **independently** from other cards, and that the random experiment continues for an arbitrary number of steps.

   Design an efficient algorithm that, given the initial integer $n \in \{1, 2, \ldots, 10\}$ picked by the volunteer and a natural number p, computes the probability that the card in position p is one of their focus cards. Give pseudocode for your algorithm and analyze its running time as a function of p using big-O notation. (The convention you use for your pseudocode is up to you, but make sure you specify the input and the output clearly; keep it fairly high-level.)

3* **(Optional, no collaboration)** Consider the same random experiment as in Problem 2.

   Design an efficient algorithm that, given the initial distinct integers $n_1, n_2 \in \{1, 2, \ldots, 10\}$ picked by two volunteers and a natural number p, computes the probability that the card in position p is one of the focus cards for both volunteers. Give pseudocode for your algorithm and analyze its running time as a function of p using big-O notation.