Homework 10 – Due Thursday, April 12, 2018 at 4:45pm

Submit solutions to all problems on separate sheets. They will be graded by different people.

Page limit  You can submit at most 1 sheet of paper per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder  Collaboration is permitted, but you must write the solutions by yourself without assistance, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises  Please practice on exercises in Chapter 5 of Mitzenmacher-Upfal.

Problems

1. (Expectations in random graphs) Let $G$ be a random graph generated from $G_{n,p}$. For each of the following parts, express $p$ as a function $n$, using asymptotic notation.

   (a) Find $p$, so that the expected number of 4-cliques in $G$ is 1. (Recall that a $k$-clique is a complete graph on $k$ nodes.)

   (b) Find $p$, so that the expected number of isolated vertices in $G$ is 1.

   (c) Find $p$, so that the expected number of Hamiltonian cycles in $G$ is 1.

    Hint: Observe that $(v_1,v_2,v_3,v_4), (v_2,v_3,v_4,v_1)$, and $(v_4,v_3,v_2,v_1)$ are different representations of the same cycle.

   Now consider a graph $G$ on $n$ nodes obtained as follows: we start with a graph with no edges and then add one edge at a time, chosen uniformly from all edges not currently in $G$, until $G$ becomes connected. Let $X$ be the number of edges in $G$. Your goal is to compute an upper bound on $E[X]$, following the guidelines below.

   (d) Express $X$ as a sum of $n-1$ random variables. To do this, break down the random process into epochs, as we did in the analysis of Coupon Collector’s problem, where the $k$-th epoch ends with a success event (that you have to define).

    Show that the probability of a success event in the $k$-th epoch is at least $\frac{k-1}{n-1}$.

   (e) Show that the expectation of a success event in the $k$-th epoch is at least $\frac{k-1}{n-1}$.

   (f) Show that the expectation of $X$ is at most $n \ln n$.

2. (Bloom filters)

   (a) Exercise 5.23.

   (b) Exercise 5.25.

3. (Finding Hamiltonian cycles) Consider the algorithm for finding Hamiltonian cycles covered in class (Algorithm 5.2 in MU).
(a) Hamiltonian cycles in directed graphs can be defined analogously to Hamiltonian cycles in undirected graphs. However, the algorithm we discussed does not work for finding Hamiltonian cycles in directed graphs. Explain why.

In the next two parts, find suitable randomized strategies for placing edges in the adjacency lists so that conditions of Theorem 5.17 are satisfied.

(b) Moved to the optional problem.

(c) Explain how to apply Algorithm 5.2 when $G$ is chosen according to $G_{n,M}$ instead of $G_{n,p}$ and the number of edges $M \geq cn \ln n$ for a sufficiently large $c$.

Hint: A graph from $G_{n,p}$ can be generated by first choosing the number of edges $X$ (according to which distribution?) and then generating a graph from $G_{n,X}$.

4* (Optional) MU explains how to apply Algorithm 5.2 when $p$ is known (and sufficiently large). Explain how to apply it when $p$ is not known (but sufficiently large).