Homework 11 – Due Friday, April 17, 2020 by noon on Gradescope.

Page limit You can submit at most 1 sheet of paper per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder Collaboration is permitted, but you must write the solutions by yourself without assistance, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises in Chapter 6 of Mitzenmacher-Upfal.

Problems

1. (Sample and Modify)

(a) Let $m, n$ and $k$ be natural numbers satisfying

$$m = n - \binom{n}{k} \cdot \frac{1}{3^{\binom{k}{2}} - 1}.$$ 

Prove that it is possible to 3-color the edges of $K_m$ so that there is no monochromatic clique of size $k$.

*Hint:* Start by 3-coloring the edges of $K_n$, then fix things up.

(b) Given a graph $G$, a subset of vertices $S$ is a called a dominating set if, for each vertex $v$, either $v \in S$ or $v$ has some neighbor in $S$.

Let $G$ be an $n$-node graph with smallest degree equal to $d$. Show that $G$ contains a dominating set of size at most

$$\frac{n(\ln(d + 1) + 1)}{d + 1}.$$ 

*Hint:* Sample vertices with probability $p = \frac{\ln(d + 1)}{d + 1}$ and then fix things up.

2. (Isolated vertices) Consider a graph $G$ generated according to $G_{n,p}$ with $p = \frac{c \ln n}{n}$ for some constant $c$. Let $p^+$ be the probability of the event that $G$ has at least one isolated vertex.

(a) Suppose $c > 1$. Show that $p^+ \to 0$ as $n \to \infty$.

(b) Suppose $c < 1$. Use the conditional expectation inequality to show that $p^+ \to 1$ as $n \to \infty$.

3. (Midterm-substitution problem, no collaboration allowed) From the moment you read this problem until you complete your solution, please do not search the internet for anything related to this problem.

The full version of this assignment, including this problem, will be posted by Monday on piazza under “Resources”.

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