Homework 11 – Due Thursday, November 21, 2024.

Page limit You can submit **at most** 2 pages per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises in Chapter 6 of Mitzenmacher-Upfal.

Problems

1. (**Tournaments**) Exercise 6.9.

2. (**Sample and Modify**)

   (a) Let m, n and k be natural numbers satisfying
   $$m = n - \binom{n}{k} \cdot \frac{1}{3^{\binom{k}{2}-1}}.$$
   Prove that it is possible to 3-color the edges of K_m so that there is no monochromatic clique of size k.

   Hint: Start by 3-coloring the edges of K_n, then fix things up.

   (b) Given a graph G, a subset of vertices S is a called a *dominating set* if, for each vertex v, either $v \in S$ or v has some neighbor in S.

   Let G be an n-node graph with smallest degree equal to d. Show that G contains a dominating set of size at most
   $$\frac{n(\ln(d+1)+1)}{d+1}.$$
   Hint: Sample vertices with probability $p = \frac{\ln(d+1)}{d+1}$ and then fix things up.

3. (**Isolated vertices**) Consider a graph G generated according to $G_{n,p}$ with $p = \frac{c \cdot \ln n}{n}$ for some constant c. Let p^+ be the probability of the event that G has has at least one isolated vertex.

   (a) Suppose $c > 1$. Show that $p^+ \to 0$ as $n \to \infty$.

   (b) Suppose $c < 1$. Use the conditional expectation inequality to show that $p^+ \to 1$ as $n \to \infty$.

4. (**Vertex Coloring, 5 points**)

   Let $G = (V, E)$ be an undirected graph. Suppose each vertex $v \in V$ is associated with a set of $S(v)$ of $6r$ colors, where $r \geq 1$. Suppose, in addition, that for each $v \in V$ and $c \in S(v)$, there are at most r neighbors u of v such that c lies in $S(u)$. Prove that there is a proper coloring of G assigning to each vertex v a color from its class $S(v)$ such that , for each edge $(u, v) \in E$, the colors assigned to u and v are different.

Hints: (1) Consider the events $B_{u,v,c}$ that u and v are both colored with color c (for different vertices $u, v \in V$ and color c).
(2) For the result you decide to use, consider the version from the lecture slides (not from the book).

5*. (**Optional, no collaboration**) Let $n \in \mathbb{N}$ and $r, \delta \in (0,1)$. A *binary code* $\mathcal{C}$ of length n is a set of n-bit strings. The code has *rate* r if the size of the set is 2^{rn}. The (relative Hamming) *distance* between two length-n strings x and y is the fraction of bits on which they differ, i.e., $\Pr_{i\in[n]}[x_i \neq y_i]$. The *distance* δ of a code $\mathcal{C}$ is the smallest distance between two strings in the code.

For example, the code

$$\mathcal{C} = \{\texttt{0000000}, \texttt{0001110}, \texttt{0010101}, \texttt{0011011}, \texttt{0100011}, \texttt{0101101}, \texttt{0110110}, \texttt{0111000}, \\ \texttt{1000111}, \texttt{1001001}, \texttt{1010010}, \texttt{1011100}, \texttt{1100100}, \texttt{1101010}, \texttt{1110001}, \texttt{1111111}\}$$

has length $n = 7$, rate $r = 4/7$, and distance $\delta = 3/7$.

Use the probabilistic method to show that for every $\delta \in (0, 1/2)$, there exists a constant $r > 0$ such that for sufficiently large n, there exists a code $\mathcal{C}$ of length n, distance at least δ, and rate r.