CS 537: *Randomness in Computing* — Prof. Sofya Raskhodnikova
Boston University — December 5, 2024

Homework 12 – Due Thursday, December 5, 2024.

Page limit You can submit **at most** 2 pages per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises in Chapter 7 of Mitzenmacher-Upfal.

Problems

1. (**Algorithmic LLL**) Show how, given integers k and n, to find, in time $n^{O(k)}$ and with high probability, a coloring of the edges of the complete graph K_n with 3 colors such that no k-clique is monochromatic (that is, has all its edges colored the same way) as long as $8\binom{k}{2}\binom{n-2}{k-2}3^{1-\binom{k}{2}} \leq 1$.

   Hint: Use (with some modifications) the algorithmic version of the Lovasz Local Lemma.

2. (**Building on the 2-SAT algorithm**) A *coloring* of a graph G is an assignment of a color to each of its vertices. If $k \geq 2$ is an integer, then k*-coloring* of G is a coloring of G with k colors such that **no** two adjacent vertices have the same color. A graph G is k*-colorable* if there exists a k-coloring of G. (For example, 2-colorable is the same as bipartite.)

   Let G be a 3-colorable graph. Consider the following algorithm for coloring the vertices of G with 2 colors so that no triangle is monochromatic. *Start with an arbitrary coloring of vertices in G with 2 colors. While there exists a monochromatic triangle in G, pick any such triangle T and change the color of uniformly random vertex of T.*

   (a) Derive an upper bound on the expected number of such recoloring steps needed for the algorithm to find a coloring with 2 colors with no monochromatic triangles. Your upper bound should be polynomial in the number of vertices in G.

   Hint: Let C be a 3-coloring of G with "colors" in $\{0, 1, 2\}$. Let U be the set of vertices in G assigned colors 0 or 1 by C. Keep track of the number of vertices in U whose colors are the same as those assigned by C.

   (b) If you only used the fact that $|U| \leq n$ in the previous part, notice that we can permute the names of the colors in the 3-coloring C. Use this to slightly improve your bound in (b).

   (c) Now we change the algorithm to start with a uniformly random assignment of two colors to vertices (instead of an arbitrary one). How does it affect the expected number of recoloring steps?

3. (**Markov Chains**)

   (a) Exercise 7.1 (a,b,c).

   (b) Exercise 7.7.

4. (**Lopsided Tipsy**) In class, we studied the Drunkard's problem, where Tipsy was equally likely to go left or right at each time step. Consider the case when his walk is lopsided: he is more likely to go left than right. Suppose that Tipsy starts at position i, his home is at position n, and the river is at position 0. At each time step, Tipsy's position decreases by 1 with probability 3/4 and increases by 1 with probability 1/4. Directions taken at different steps are mutually independent. Tipsy's walk finishes when he reaches his house or the river.

   (a) Let D_t be the net distance towards home Tipsy has traveled after taking t steps. (If he moved to the left on the first step, D_1 is negative.) Prove that $\mathbb{E}[3^{D_{t+1}}] = \mathbb{E}[3^{D_t}]$.

   (*Hint:* Use the Law of Total Expectation.)

   (b) Determine the probability that Tipsy eventually ends up in the river and the probability that he ends up home. (*Hint:* Consider $\lim_{t\to\infty} \mathbb{E}[3^{D_t}]$ and use the definition of expectation. Note that as $t \to \infty$, Tipsy will end up either home or in the river.)

   (c) Generalize the argument in previous parts to the case where Tipsy goes left with probability $p > \frac{1}{2}$.

   (*Hint:* Consider $\mathbb{E}[c^{D_t}]$ for some c that depends on p.)

5*. (**Optional, no collaboration**) Design a problem you would put on the final for CS 537 if you were teaching the course. Preferably the problem should be on one of the algorithms we covered in the course, not just on probability.

   Hint: If your problem is so good that it gets selected for the final exam, you will know how to solve one problem on the final in advance.