Homework 2 – Due Thursday, February 1, 2018 at 5:00pm

Submit solutions to problems 1-3 on separate sheets. They will be graded by different people.

**Page limit** You can submit at most 1 sheet of paper per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

**Reminder** Collaboration is permitted, but you must write the solutions by yourself without assistance, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

**Exercises** Please practice on exercises in Chapter 1 of Mitzenmacher-Upfal.

**Problems**

0. (**Nameplate**) Please print out a nameplate with your name and bring it to every lecture. A template is available at the bottom of the course web page.

1. (**Random band**)  

   (a) In a group of ten people, seven can play the piano, five can play the saxophone, four can play the violin, four can play the piano and the saxophone, three can play the piano and the violin, two can play the saxophone and the violin, and one person can play all three instruments. Suppose a person is picked uniformly at random from the group. Use the inclusion-exclusion principle to calculate the probability that this person can play at least one instrument. Explain your calculation clearly.

   Upon hearing that the band practiced in MCS 148 and put out two pizzas there, you rush to the building to get in on the action. Assume that:

   - it is equally probable that 2, 1, or none of the pizzas are left by the time you get to MCS 148 (1/3 probability each);
   - if both pizzas are left, the probability of your smelling pizza when you first walk into the building is 1; if only one is left, it’s 2/3; and if they’re gone, it’s 1/3 (the smell might still linger).

   Compute the following probabilities, explaining your calculation in each case.

   (b) What is the probability that there is no pizza left, but that you’ll still smell pizza when you walk in?
   (c) What is the probability that you’ll smell pizza when you walk in?
   (d) As you walk into MCS, you smell pizza. What is the probability that there is at least one pizza left?
2. (Chess board) Consider an $8 \times 8$ chess board in which the rows are numbered from 1 to 8, and likewise for the columns. And, as is usual for a chess board, the squares are alternately colored black and white. The squares of this chess board form the elements of a sample space in which all of the 64 squares on the chess board are equally likely; that is, all have probability $\frac{1}{64}$.

For each of the following pairs of events $A$ and $B$, determine if the two events are independent and prove your answer is correct:

(a) $A$ is the event that a white square is chosen and $B$ is the event that a black square is chosen.

(b) $A$ is the event that a square from an even numbered row is chosen and $B$ is the event that a square from an even numbered column is chosen.

(c) $A$ is the event that a white square is chosen and $B$ is the event that a square from an even numbered column is chosen.

Determine if the following three events $A$, $B$, and $C$ are mutually independent and prove your answer is correct:

(d) $A$ is the event that a square from an even numbered row is chosen, $B$ is the event that a square from an even numbered column is chosen, and $C$ is the event that a white square is chosen.

3. (Marble bag) You are given a bag containing ten marbles and are told that the number of red marbles in the bag is equally likely to be any number between 0 and 10 inclusively; the other marbles are green. So, in particular, the probability that all ten marbles are red is $\frac{1}{11}$.

(a) Suppose you pick a marble at random from the bag and it is red. Find the probability that all the marbles in the bag are red, given that we have picked one red marble. In other words, we want to compute the conditional probability $\Pr[A|R]$, where $R$ is the event that the chosen marble is red, and $A$ is the event that all marbles in the bag are red. Hint: Use Bayes’ law and the law of total probability.

(b) Is the probability $\Pr[A|R]$ larger or smaller than $\Pr[A]$? Can you explain why it is reasonable?

4. (Multiple-choice test) Adam and Bella are taking a multiple choice test, where each question has $m$ choices.

(a) For each question, Adam either knows the answer or has no clue. If he has no clue, he picks one of the $m$ choices uniformly at random. Let $p$ be the probability that Adam knows the answer. What is the conditional probability that Adam knows the answer to a question, given that he answered it correctly? Express your answer in terms of $m$ and $p$.

(b) Evaluate the expression you got in part (a) for $m = 5$ and $p = .6$.

(c) For each question, Bella either (i) knows the answer, (ii) can eliminate all but 2 answers, or (iii) has no clue. If she can eliminate all but 2 answers, she pick among the remaining 2 answers uniformly at random. If she has no clue, she picks one of the $m$ choices uniformly at random. Let $p_1$ be the probability that Bella knows the answer and $p_2$ be the probability that she can eliminate all but 2 answers. What is the conditional probability that Bella knows the answer to a question, given that she answered it correctly? Express your answer in terms of $m, p_1$ and $p_2$.

(d) Suppose $m = 5$ and $p_2 = .1$. If your expression in part (c) evaluates to the same value as what you got in part (b), what is $p_1$? (Think, but do not hand in: Is it higher or lower than .6? Can you explain why?)
5* \textbf{(Optional, no collaboration)} Professor Sofya offers you a chance to get an A++ in the course by playing a simple game. She gives you 50 cookies with white chocolate chips, 50 cookies with dark chocolate chips, and 2 empty identical containers. She asks you to put these 100 cookies into the two containers. For each cookie, you must decide which container to put it in. Then you will be blindfolded, and the positions of the containers will be mixed up. After that, you can choose one container and take ONE cookie. (All cookies feel identical to the touch, and you are not allowed to modify them in any way.) If the cookie you took has dark chocolate chips, you get an A++ for the course. Otherwise, you have to do all homework assignments and take three exams to earn at most A.

Figure out which cookies to put in each container so that you have the greatest probability of choosing a cookie with dark chocolate chips.