Homework 3 – Due Thursday, February 8, 2018 at 4:45pm

Submit solutions to all problems on separate sheets. They will be graded by different people.

Page limit  You can submit at most 1 sheet of paper per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder  Collaboration is permitted, but you must write the solutions by yourself without assistance, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises  Please practice on exercises in Chapter 1 of Mitzenmacher-Upfal.

Problems

1. (Random subsets)

   (a) We generate a subset $X$ of set $[n]$ as follows: a fair coin is flipped independently for each element of the set; if the coin lands heads then the element is added to $X$, and otherwise it is not. Argue that the resulting set $X$ is equally likely to be any one of the $2^n$ possible subsets.

   (b) Suppose that two sets $X$ and $Y$ are chosen independently and uniformly at random from all the $2^n$ subsets of $[n]$. Determine $\Pr[X \subseteq Y]$. Hint: Use part (a).

   (c) Suppose that three sets $X, Y$ and $Z$ are chosen independently and uniformly at random from all the $2^n$ subsets of $[n]$. Determine $\Pr[X \cup Y = Z]$.

2. (Verifying matrix multiplication)

   (a) In our algorithm for verifying matrix multiplication, we chose a uniformly random vector $\bar{r}$ from $\{0, 1\}^n$. Suppose we choose $\bar{r}$ from $\{0, 1, \ldots, v-1\}^n$ instead. Prove a better bound (than $1/2$) on the probability of success of one check performed by the algorithm.

   (b) Suppose your prior belief is that a given matrix multiplication identity is correct with probability $1/2$. The analysis that starts at the bottom of page 11 of the textbook demonstrates how your prior belief about the probability that the identity is correct changes after $i$ runs of the algorithm that do not find any mistakes. Modify this analysis to work with your bound from part (a).

3. (Improved Randomized Min-Cut Algorithm) Exercise 1.25 in the book.

4. (Detecting defects) You are in charge of inspecting cookies baked in your company. A worker is unreliable if the proportion of defective cookies he bakes is $\alpha$ or higher. (For example, if he bakes 1000 cookies in a day, $\alpha1000$ of them will be defective.)

   (a) You would like to make sure that if a worker is unreliable, you will find at least one defective cookie with probability 99% and fire him. You do not have time to inspect every single cookie. Instead you decide to choose $k$ cookies uniformly at random. For simplicity, let’s say that you
will choose cookies with replacement. That is, on each try, you find a defective cookie with probability $p$, where $p$ is the proportion of defective cookies the worker baked. Prove that if

$$k \geq \frac{\ln 100}{\alpha}$$

and $p \geq \alpha$ than you will find at least one defective cookie with probability 99%.

*Hint:* Use the fact that $1 + x \leq e^x$.

(b) You got a raise, and now you are inspecting cookies made by $n$ workers. You would like to ensure that with probability at least 99%, all unreliable workers will be fired, that is, you will detect a defective cookie for every one of them. You use the same random strategy as in part (a) for each worker, but now you have to inspect more cookies per worker because you want to achieve overall success probability of 99%, not just per worker. What should you choose $k$ to be as a function of $\alpha$ and $n$? Find the smallest $k$ you can.

*Hint:* Use the union bound.