Homework 5 – Due Thursday, October 7, 2021

Page limit You can submit at most 2 pages per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder Collaboration is permitted, but you must write the solutions by yourself without assistance, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises in Chapter 3 of Mitzenmacher-Upfal.

Problems

1. (Random hats) Suppose $n$ people come to a theater performance wearing hats. When they leave, they get back a uniformly random hat. That is, each of $n!$ assignments of $n$ hats to $n$ people is equally likely.

   (a) We say that a pair of people (let’s call them Alice and Bob) exchanged their hats if Alice got Bob’s hat and Bob got Alice’s hat. Let $X$ denote the number of pairs of people that exchanged their hats. Calculate the expectation of $X$.

   Hint: Write $X$ as a sum of indicator random variables.

   (b) Calculate the variance of $X$.

   Hint: Use the same sum as in part (a); remember that the indicators are not independent.

2. (Random counter, part 2) This is a continuation of problem 4 from homework 3. You do not need to recompute the quantities we obtained in that problem and may directly use them in your computations.

   (a) Let $\tilde{m}$ be the value of $2^X - 1$ after the door has been opened $m$ times. Fix a parameter $\epsilon > 0$. We say that $\tilde{m}$ is a $(1 \pm \epsilon)$-approximation of $m$ if

   $$(1 - \epsilon)m \leq \tilde{m} \leq (1 + \epsilon)m.$$ 

   Use Chebyshev’s inequality to give an upper bound in terms of $\epsilon$ on the probability that $\tilde{m}$ is NOT a $(1 \pm \epsilon)$-approximation of $m$.

   (b) To decrease the error probability, you decide to keep $t$ independent random counters. That is, every time the door opens, each counter $X_j$ for $j \in [t]$ is incremented with probability $2^{-X_j}$, and the random coins used for different counters are mutually independent. Your new estimate is the average of the estimates you get from the counters: $\tilde{m} = \frac{1}{t} \sum_{j \in [t]} (2^{X_j} - 1)$.

   What is the expectation and variance of the new estimate?

   (c) Redo part (a) for the new estimate to get a bound in terms of $\epsilon$ and $t$. What should $t$ be set to as a function of $\epsilon$ to ensure that $\tilde{m}$ is a $(1 \pm \epsilon)$-approximation of $m$ with probability at least 99%?
3. (Generalization of Randomized Median Algorithm) In this problem, you are asked to generalize the randomized median-finding algorithm from class (Section 3.5 of the MU book), so that it finds an element of rank \( k \) (that is, the \( k \)th smallest element) in an array of \( n \) distinct elements, for any given \( k \in [4n^{3/4}, n - 4n^{3/4}] \). You may ignore rounding issues in your algorithm and analysis.

(a) Explain how to modify lines 3, 4, 6 and 8 of the algorithm in the book.
(b) Analyze the running time of the modified algorithm.
(c) We will follow the same analysis outline as in class (and in the book). Change the definitions of events \( \mathcal{E}_1, \mathcal{E}_2, \mathcal{E}_{3,1} \) and \( \mathcal{E}_{3,2} \), so that they apply for general \( k \).
(d) Write \( \Pr[\mathcal{E}_1] \) and \( \Pr[\mathcal{E}_{3,1}] \) as probability expressions involving the tail of a suitable binomial random variable.

Do not repeat the rest of the analysis (which is essentially the same as in the case of finding the median).

4* (Optional, just for fun, don’t hand in, but feel free to discuss with us) This problem is in response to a student question about generating random bits.

(a) Suppose you are given a biased coin with probability of HEADS equal to \( p \), but you don’t know \( p \). Assume that \( p \notin \{0, 1\} \). Devise a strategy for generating a uniformly random bit by tossing the biased coin multiple times. The expected number of coin tosses needed should be at most \( \frac{1}{p(1-p)} \).

Hint: Consider two consecutive coin tosses.

(b) Now try to generalize your strategy to generate as many independent and uniform bits as you can from a given number of tosses of the biased coin.