Homework 5 – Due Thursday, February 22, 2018 at 4:45pm

Submit solutions to all problems on separate sheets. They will be graded by different people.

**Page limit** You can submit at most 1 sheet of paper per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

**Reminder** Collaboration is permitted, but you must write the solutions by yourself without assistance, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

**Exercises** Please practice on exercises in Chapter 3 of Mitzenmacher-Upfal.

**Problems**

1. *(Selling a plane)* You built a plane and would like to sell it to the buyer who makes the best offer. You received responses from \( n \) potential buyers who asked to see the plane. They will come to your factory one at a time and name their price. Each insists on an immediate decision from you whether you accept their price or not. You can compare each offer to all the previous offers (assume there are no ties), but you don’t know what subsequent buyers will offer. Assume that buyers come in a random order, chosen uniformly at random from all \( n! \) possible orderings. You want to maximize the probability of choosing the best offer.

   Consider the following strategy. You do not accept any of the first \( m \) offers, only collect information about how much buyers value your plane. After seeing buyer \( m \), you change your strategy: you accept the first offer that is better than all the previous ones. Your goal in this problem (broken down into parts) is to find the best value of \( m \) as a function of \( n \).

   (a) Suppose \( m < i \leq n \). What is the probability that the best among the first \( i - 1 \) offers is in the first \( m \) offers?

   (b) Let \( B \) be the event that you choose the best offer. Let \( B_i \) be the event that offer \( i \) is the best and you choose it. Determine \( \Pr[B_i] \).

   (c) Show that \( \Pr[B] = \frac{m}{n} \sum_{i=m+1}^{n} \frac{1}{i-1} \).

   (d) Show that \( \frac{m}{n} \ln \frac{n}{m} \leq \Pr[B] \leq \frac{m}{n} \ln \frac{n-1}{m-1} \).

   (e) Show that \( \frac{m}{n} \ln \frac{n}{m} \) is maximized when \( \frac{m}{n} = \frac{1}{e} \). Give a lower bound on \( \Pr[B] \) when \( m = \frac{n}{e} \).

2. *(Random hats)* Suppose \( n \) people come to a theater performance wearing hats. When they leave, they get back a uniformly random hat. That is, each of \( n! \) assignments of \( n \) hats to \( n \) people is equally likely.

   (a) We say that a pair of people (let’s call them Alice and Bob) exchanged their hats if Alice got Bob’s hat and Bob got Alice’s hat. Let \( X \) denote the number of pairs of people that exchanged their hats. Calculate the expectation of \( X \).

   *Hint:* Write \( X \) as a sum of indicator random variables.
(b) Calculate $\text{Var}[X]$.

*Hint:* Use the same sum as in part (a); remember that the indicators are not independent.

3. (**MaxCut**) In the problem MaxCut, we are given an undirected graph $G = (V, E)$ and asked to find a cut of maximum size in $G$. (Recall that a cut in $G$ is a partition of the vertex set $V$ into two parts; the size of the cut is the number of edges with one endpoint in each part of the partition.) In contrast to the seemingly very similar problem MinCut discussed in class (recall Karger’s algorithm), MaxCut is a famous NP-hard problem, so we do not expect to find an efficient algorithm that solves it exactly. Here is a very simple linear-time randomized algorithm that gives a pretty good approximation:

- Randomly and independently color each vertex $v \in V$ red or blue with probability $1/2$.
- Output the cut defined by the red/blue partition of vertices.

(a) Let random variable $X$ denote the size of the cut output by the algorithm. Compute $E[X]$ as a function of the number of edges in $G$, and deduce that $E[X] \geq OPT$, where $OPT$ is the size of a maximum cut in $G$.

*Hint:* Write $X$ as a sum of indicator random variables.

(b) Let $p$ denote the probability that the cut output by the algorithm has size at least $0.49OPT$. Show that $p \geq 1/51$.

*Hint:* Applying Markov’s inequality to $X$ will not work here. Try applying Markov’s inequality to a different random variable.

(c) Now compute the variance $\text{Var}[X]$.

*Hint:* Again write $X$ as the sum of indicators, as in part (a).

(d) Let $p$ be the probability defined in part (b). Use Chebyshev’s inequality together with part (c) to show that $p = 1 - O(1/|E|)$.

(Note how Chebyshev’s inequality gives us a better bound here than Markov’s.)