Prof. Sofya Raskhodnikova

Homework 9 – Due Thursday, November 7, 2024 on Gradescope.

Page limit You can submit **at most** 2 pages per problem (unless specified otherwise), even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder Collaboration is permitted, but you must write the solutions *by yourself without assistance*, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises in Chapter 5 of Mitzenmacher-Upfal.

Problems

1. (**From randomized to deterministic algorithms, 5 points**) Let c and k be positive integers. You have the code for a program to compute your favorite function. The program uses $c \log n$ random bits and runs in time $O(n^k)$ on inputs of length n, where c and k are constants. (The only steps in the program that use randomness are calls to function `rand`, each of which returns one random bit.) On every input, it returns the correct answer with probability at least 2/3. Explain how to obtain a deterministic polynomial-time program for computing this function. Analyze its running time.

2. (**Expectations in random graphs, 15 points, page limit of 3 pages**) Let G be a random graph generated from $G_{n,p}$. For each of the following parts, express p as a function n, using asymptotic notation.

   (a) Find p, so that the expected number of 4-cliques in G is 1. (Recall that a k-clique is a complete graph on k nodes.)

   (b) Find p, so that the expected number of isolated vertices in G is 1.

   (c) Find p, so that the expected number of Hamiltonian cycles in G is 1.
   Hint: Observe that $(v_1, v_2, v_3, v_4), (v_2, v_3, v_4, v_1)$, and (v_4, v_3, v_2, v_1) are different representations of the same cycle.

   Now consider a graph G on n nodes obtained as follows: we start with a graph with no edges and then add one edge at a time, chosen uniformly from all edges not currently in G, until G becomes connected. Let X be the number of edges in G. Your goal is to compute an upper bound on $\mathbb{E}[X]$, following the guidelines below.

   (d) Express X as a sum of $n-1$ random variables. To do this, break down the random process into epochs, as we did in the analysis of Coupon Collector's problem, where the k-th epoch ends with a success event (that you have to define).
   Decide how to number the epochs and show that the probability of a success event in the k-th epoch is at least $\frac{k-1}{n-1}$.
   Hint: The epochs don't have to be numbered from 1 to $n-1$. Choose what to count as the k-th epoch so that it works nicely with your definition of success.

(e) Show that the expectation of X is at most $n \ln n + O(n)$.

3. (**Finding Hamiltonian cycles**) Consider the algorithm for finding Hamiltonian cycles (Algorithm 5.2 in MU).

   (a) Hamiltonian cycles in directed graphs can be defined analogously to Hamiltonian cycles in undirected graphs. However, the algorithm in the book does not work for finding Hamiltonian cycles in directed graphs. Explain why.

   In the next two parts, find suitable randomized strategies for placing edges in the adjacency lists so that conditions of Theorem 5.17 are satisfied.

   (b) Explain how to apply Algorithm 5.2 when G is chosen according to $G_{n,M}$ instead of $G_{n,p}$ and the number of edges $M \geq cn \ln n$ for a sufficiently large constant c.

   Hint: A graph from $G_{n,p}$ can be generated by first choosing the number of edges X (according to which distribution?) and then generating a graph from $G_{n,X}$.

   (c) MU explains how to apply Algorithm 5.2 when p is known (and sufficiently large). Explain how to apply it when p is not known (but sufficiently large).

   Hint: Use part (b) if the graph has sufficiently many edges; otherwise, fail.