Homework 9 – Due Friday, April 3, 2020 by noon on Gradescope.

Page limit You can submit at most 1 sheet of paper per problem, even if the problem has multiple parts. If you submit a longer solution for some problem, only the first sheet of paper will be graded.

Reminder Collaboration is permitted, but you must write the solutions by yourself without assistance, and be ready to explain them orally to the instructor if asked. You must also identify your collaborators and whether you gave help, received help, or worked something out together. Getting solutions from outside sources such as the Web or students not enrolled in the class is strictly forbidden.

Exercises Please practice on exercises in Chapter 5 of Mitzenmacher-Upfal.

Problems

1. (Expectations in random graphs) Let $G$ be a random graph generated from $G_{n,p}$. For each of the following parts, express $p$ as a function $n$, using asymptotic notation.

   (a) Find $p$, so that the expected number of 4-cliques in $G$ is 1. (Recall that a $k$-clique is a complete graph on $k$ nodes.)
   (b) Find $p$, so that the expected number of isolated vertices in $G$ is 1.
   (c) Find $p$, so that the expected number of Hamiltonian cycles in $G$ is 1.

   Hint: Observe that $(v_1, v_2, v_3, v_4), (v_2, v_3, v_4, v_1)$, and $(v_4, v_3, v_2, v_1)$ are different representations of the same cycle.

Now consider a graph $G$ on $n$ nodes obtained as follows: we start with a graph with no edges and then add one edge at a time, chosen uniformly from all edges not currently in $G$, until $G$ becomes connected. Let $X$ be the number of edges in $G$. Your goal is to compute an upper bound on $E[X]$, following the guidelines below.

   (d) Express $X$ as a sum of $n - 1$ random variables. To do this, break down the random process into epochs, as we did in the analysis of Coupon Collector’s problem, where the $k$-th epoch ends with a success event (that you have to define).

   Show that the probability of a success event in the $k$-th epoch is at least $\frac{k - 1}{n - 1}$.

   (e) Show that the expectation of $X$ is at most $n \ln n$.

2. (Using hashing for comparing multisets) Exercise 15.11.

3. (Bloom filters)

   (a) Exercise 5.23.
   (b) Exercise 5.25.

4. (Midterm-substitution problem, no collaboration allowed) From the moment you read this problem until you complete your solution, please do not search the internet for anything related to this problem.

The full version of this assignment, including this problem, will be posted by Monday on piazza under “Resources”.