Homework 4 – Due Thursday, November 12 *before* 10am on Gradescope

Instructions

- Solutions written in LaTeX are strongly preferred, but you can upload any pdf files, including scanned hand-written solutions. Template latex files are on the course webpage.
- Collaboration is allowed and encouraged. However, each of you should think about a problem before discussing it with others and write up your solution independently. You may consult books and online sources to get information about well-known theorems, such as the Chernoff bound. But you are not allowed to look up solutions directly in papers or any other sources. And you *must* list all collaborators and sources! (See full details in the General Information Handout.)
- Correctness, clarity, and succinctness of the solution will determine your score.

Problems

1. This is a collection of questions with short answers on testers in the adjacency matrix model.
   (a) Based on our analysis in class, give a new ϵ-test for bipartiteness with query and time complexity $O(\frac{\log^2 1/\epsilon}{\epsilon^3})$ instead of $O(\frac{\log^2 1/\epsilon}{\epsilon^4})$. Provide a one-line explanation.
   (b) In class we saw an ϵ-additive approximation algorithm for the edge density of the max cut with running time exponential in $\frac{1}{\epsilon}$. Explain why it is unlikely that there is such an algorithm with $poly(\frac{1}{\epsilon})$ running time that works for all ϵ. (Hint: it would imply NP $\subseteq$ BPP.)
   (c) Improve the running time of the probabilistic algorithm (given in class) that finds a cut with density at least $\mu(G) - \epsilon$ from $2^{poly\frac{1}{\epsilon}} n^2$ to $2^{poly\frac{1}{\epsilon}} + O(n \cdot poly\frac{1}{\epsilon})$. Explain the new algorithm and give a one-line justification. Hint: we proved all the necessary lemmas in class.
2. [**Application of the Regularity Lemma**] In this problem, you are asked to show that the number of triangle-free labeled graphs on n nodes is $2^{\left(\frac{1}{4}+o(1)\right)n^2}$.
   (a) Show that there are at least $2^{\frac{1}{4}n^2}$ triangle-free labeled graphs on n nodes.
   (b) Prove (e.g., by induction) that every triangle-free graph contains at most $\frac{n^2}{4}$ edges.
   (c) Using the Regularity Lemma and 2b, show that there are at most $2^{\left(\frac{1}{4}+o(1)\right)n^2}$ triangle-free labeled graphs on n nodes.
   You may use the following facts without a proof:

$$\binom{n}{\alpha n} \leq 2^{n(H(\alpha)+o(1))} \text{ where } H(\alpha) = -\alpha \log_2 \alpha - (1-\alpha)\log_2(1-\alpha). \quad (1)$$

$$\text{The entropy function } H(\alpha) \text{ tends to 0 as } \alpha \text{ tends to 0.} \quad (2)$$