Sublinear Algorithms

Lecture 1

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Organizational

Course webpage:
https://cs-people.bu.edu/sofya/sublinear-course/

Use Piazza to ask questions
Office hours (on zoom):
Wednesdays, 1:00PM-2:30PM

Evaluation

- Homework (about 4 assignments)
- Taking lecture notes (about once per person)
- Course project and presentation
- Peer grading (PhD student only)
- Class participation

Tentative Topics

Introduction, examples and general techniques.

Sublinear-time algorithms for

- graphs
- strings
- geometric properties of images
- basic properties of functions
- algebraic properties and codes
- metric spaces
- distributions

  Tools: probability, Fourier analysis, combinatorics, codes, …

Sublinear-space algorithms: streaming

Tentative Plan

Introduction, examples and general techniques.

Lecture 1. Background. Testing properties of images and lists.

Lecture 2. (Next week) Properties of functions and graphs. Sublinear approximation.

Lecture 3-5. Background in probability. Techniques for proving hardness. Other models for sublinear computation.

Motivation for Sublinear-Time Algorithms

Massive datasets

- world-wide web
- online social networks
- genome project
- sales logs
- census data
- high-resolution images
- scientific measurements

"Why Gramma, what big data you have!"

Long access time

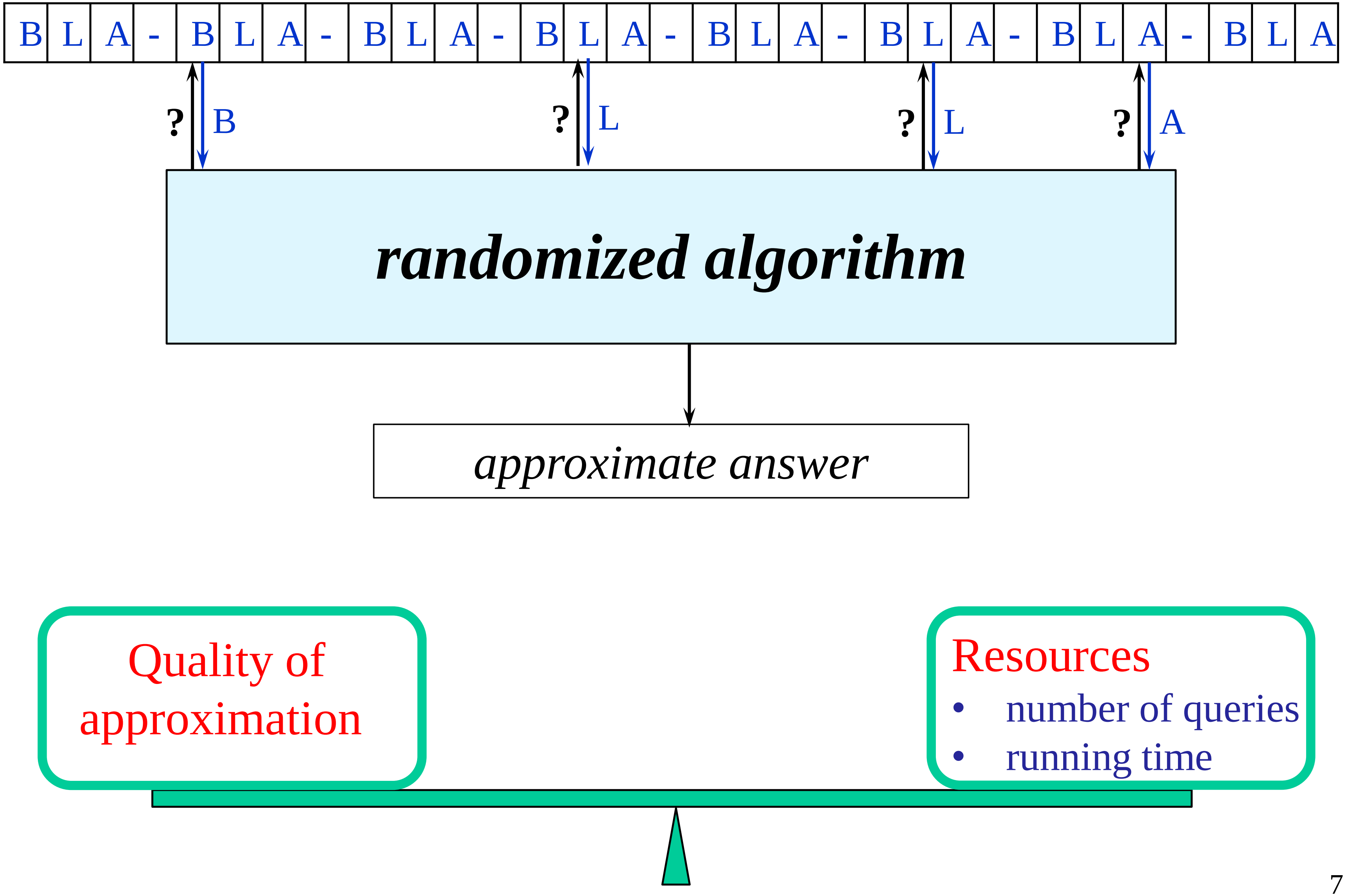
- communication bottleneck (slow connection)
- implicit data (an experiment per data point)

Do We Have To Read All the Data?

- What can an algorithm compute if it
  - reads only a **tiny** portion of the data?
  - runs in **sublinear** time?

Image source: http://apandre.wordpress.com/2011/01/16/bigdata/

A Sublinear-Time Algorithm



Goal: Fundamental Understanding of Sublinear Computation

- What computational tasks?
- How to measure quality of approximation?
- What type of access to the input?
- Can we make our computations robust (e.g., to noise or erased data)?

Types of Approximation

Classical approximation

- need to compute a value
  - output should be close to the desired value
  - example: average

Property testing

- need to answer YES or NO
  - Intuition: only require correct answers on two sets of instances that are very different from each other

Classical Approximation

A Simple Example

Approximate Diameter of a Point Set [Indyk]

Input: m points, described by a distance matrix D

- D_{ij} is the distance between points i and j
- D satisfies triangle inequality and symmetry

(Note: input size is $n = m^2$)

- Let i, j be indices that maximize D_{ij} .
- Maximum D_{ij} is the *diameter*.

Output: (k, ℓ) such that $D_{k\ell} \geq D_{ij}/2$

Algorithm and Analysis

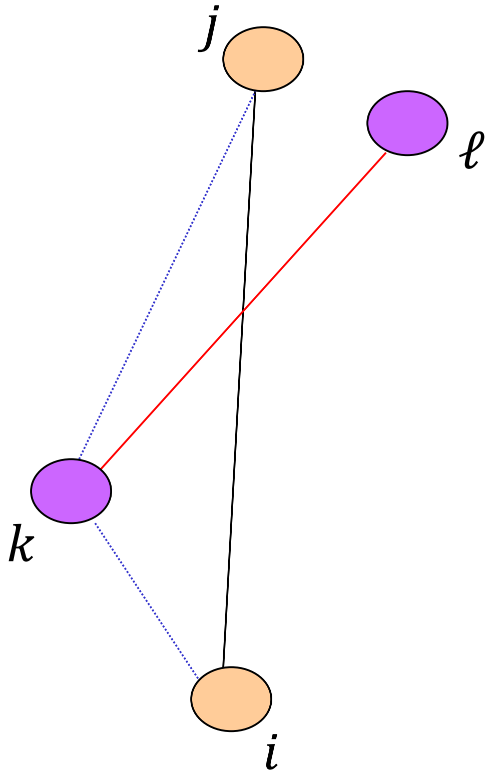
Algorithm (m, D)

1. Pick k arbitrarily
2. Pick ℓ to maximize $D_{k\ell}$
3. Output (k, ℓ)

- Approximation guarantee

$$
\begin{aligned}
D_{ij} &\leq D_{ik} + D_{kj} \quad \text{(triangle inequality)} \\
&\leq D_{k\ell} + D_{k\ell} \quad \text{(choice of } \ell \text{ + symmetry of } D\text{)} \\
&\leq 2D_{k\ell}
\end{aligned}
$$

- Running time: $O(m) = O(m = \sqrt{n})$



A rare example of a deterministic sublinear-time algorithm

Property Testing

Property Testing: YES/NO Questions

Does the input satisfy some property? (YES/NO)

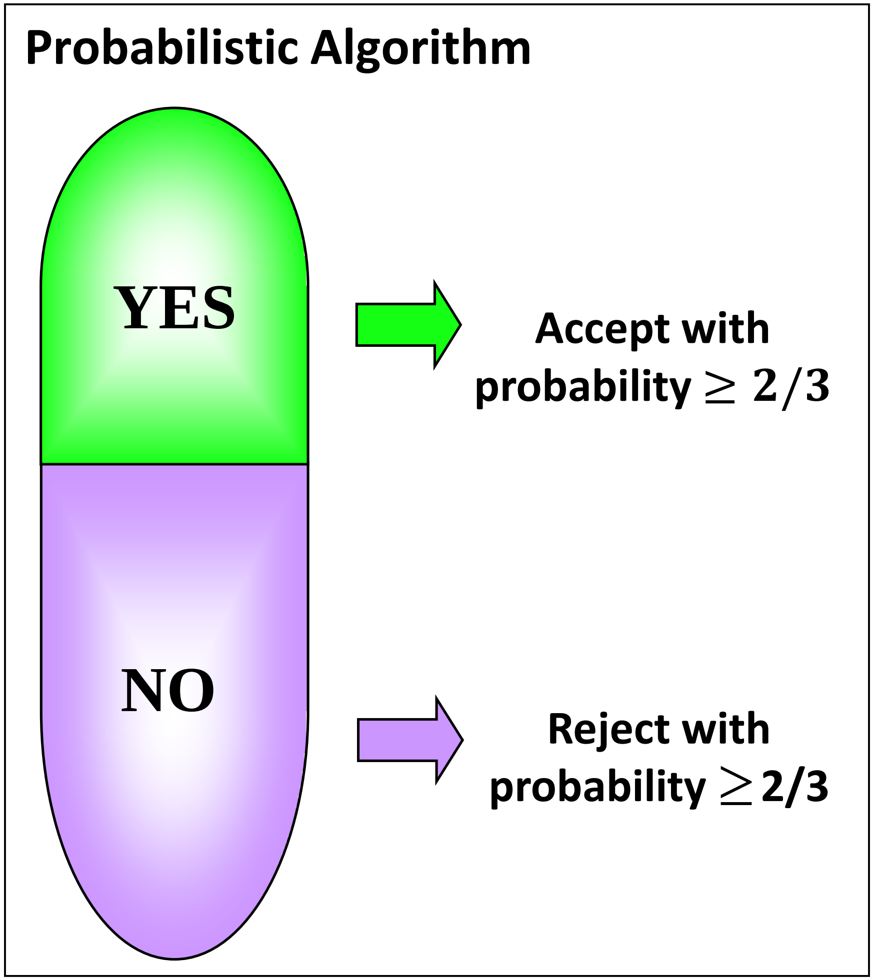
"in the ballpark" vs. "out of the ballpark"

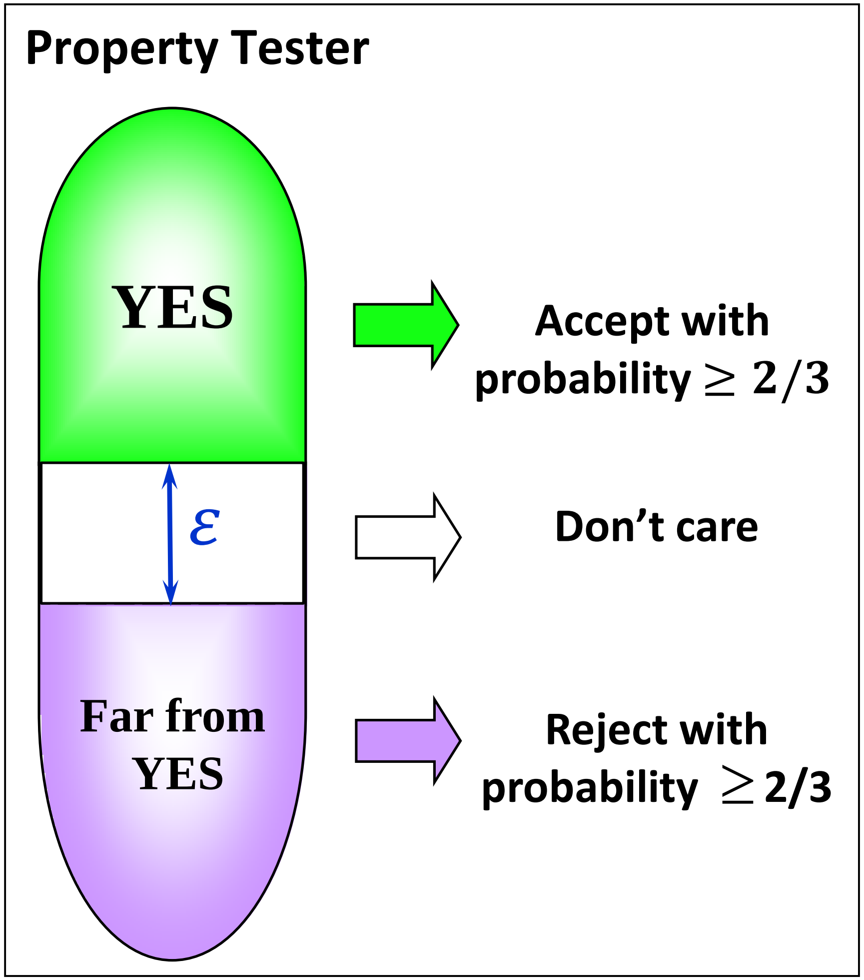
Does the input satisfy the property or is it far from satisfying it?

- **for some applications, it is the right question (probabilistically checkable proofs (PCPs), precursor to learning)**
- **good enough when the data is constantly changing**
- **fast sanity check to rule out inappropriate inputs**

(rejection-based image processing)

Property Tester Definition





ε-far = differs in many places (≥ ε fraction of places)

Randomized Sublinear Algorithms

Toy Examples

Property Testing: a Toy Example

0	0	0	1	…	0	1	0	0

Input: a string $w \in \{0,1\}^n$

Question: Is $w = 00 \ldots 0$?

Requires reading entire input.

Approximate version: Is $w = 00 \ldots 0$ or

does it have $\geq \varepsilon n$ 1's ("errors")?

Test (n, w)

1. Sample $s = 2/\varepsilon$ positions uniformly and independently at random
2. If 1 is found, **reject**; otherwise, **accept**

Analysis: If $w = 00 \ldots 0$, it is always accepted.

Used: $1 - x \leq e^{-x}$

If w is ε-far, Pr[error] = Pr[no 1's in the sample]$\leq (1-\varepsilon)^s \leq e^{-\varepsilon s} = e^{-2} < \frac{1}{3}$

Witness Lemma

If a test catches a witness with probability $\geq p$,

then $s = \frac{2}{p}$ iterations of the test catch a witness with probability $\geq 2/3$.

Randomized Approximation: a Toy Example

0	0	0	1	…	0	1	0	0

Input: a string $w \in \{0,1\}^n$

Goal: Estimate the fraction of 1's in w (like in polls)

It suffices to sample $s = 1 / \varepsilon^2$ positions and output the average to get the fraction of 1's $\pm\varepsilon$ (i.e., additive error ε) with probability $\geq$ 2/3

Hoeffding Bound

Let $Y_1, \dots, Y_s$ be independently distributed random variables in [0,1].

Let $Y = \frac{1}{s} \cdot \sum_{i=1}^{s} Y_i$ (called *sample mean*). Then $\Pr[|Y - E[Y]| \geq \varepsilon] \leq 2e^{-2s\varepsilon^2}$.

Y_i = value of sample i. Then $E[Y] = \frac{1}{s} \cdot \sum_{i=1}^{s} E[Y_i] =$ (fraction of 1's in w)

$$\Pr[|(\text{sample mean}) - (\text{fraction of } 1\text{'s in } w)| \geq \varepsilon]$$
$$\leq 2e^{-2s\varepsilon^2} = 2e^{-2} < 1/3$$

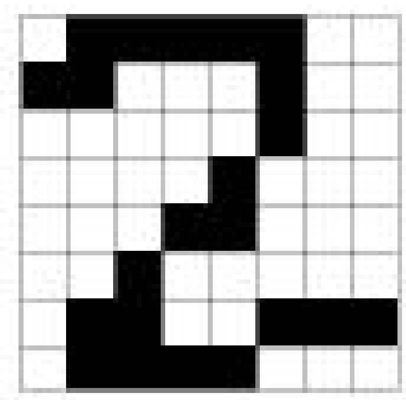
Apply Hoeffding Bound

substitute $s = 1 / \varepsilon^2$

Property Testing

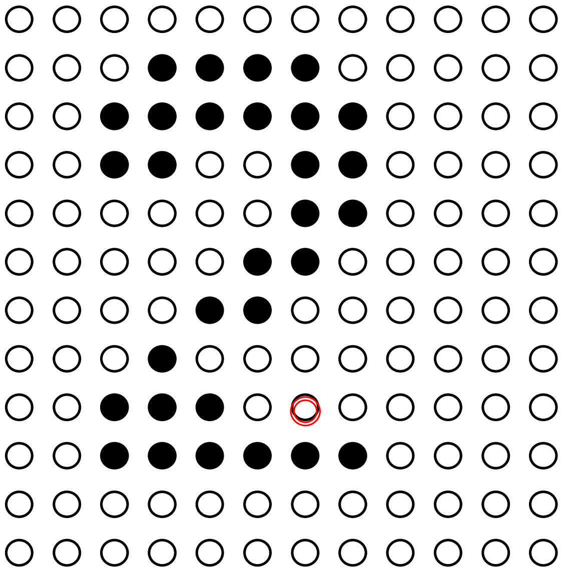
Simple Examples

Testing Properties of Images



Pixel Model

Input: $n \times n$ matrix of pixels
(0/1 values for black-and-white pictures)



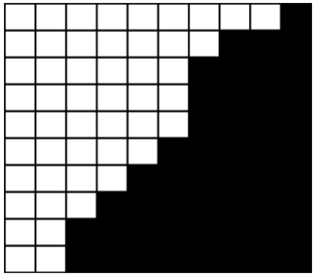
Query: point (i_1, i_2)

Answer: color of (i_1, i_2)

Testing if an Image is a Half-plane [R03]

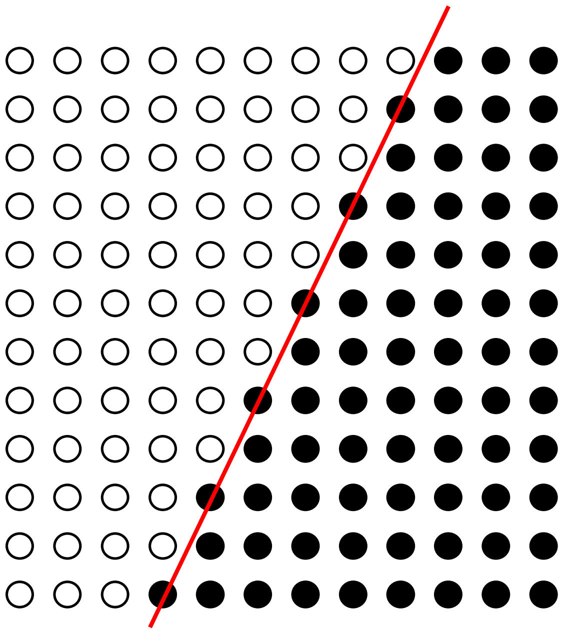
A half-plane or

ε-far from a half-plane?

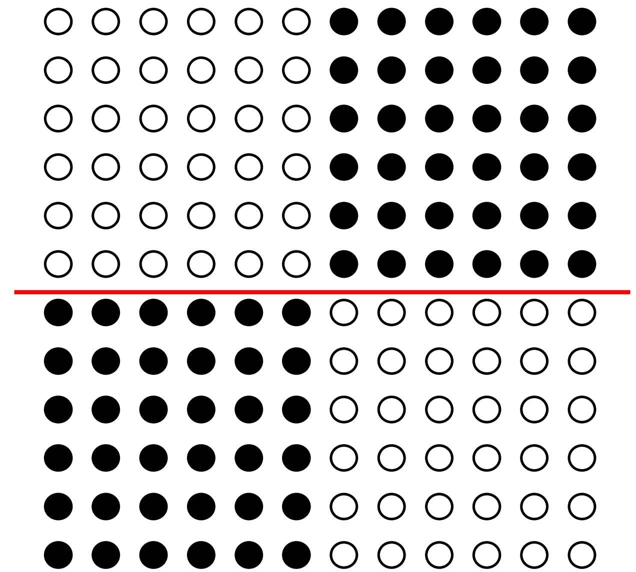


O(1/ε) time

Half-plane Instances

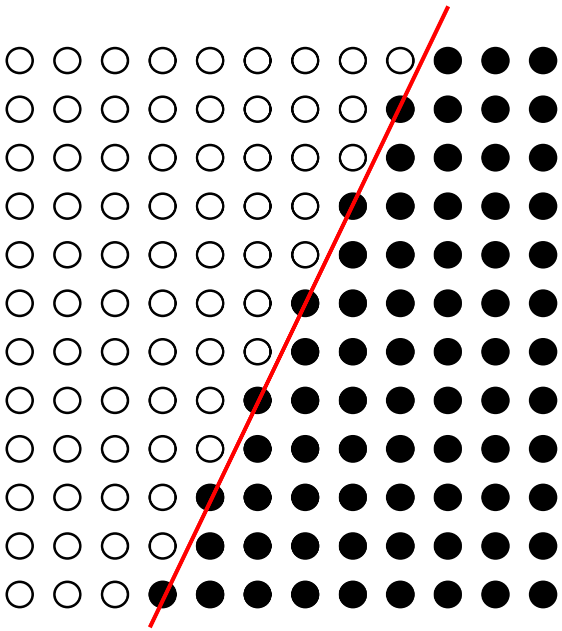


A half-plane

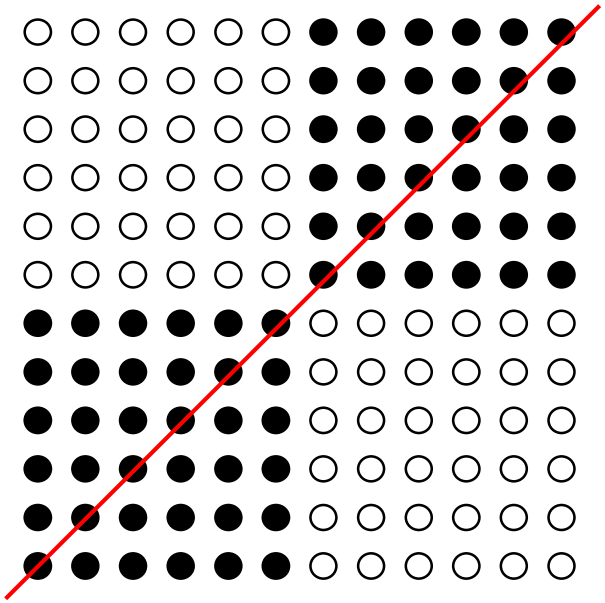


$\frac{1}{4}$-far from a half-plane

Half-plane Instances



A half-plane



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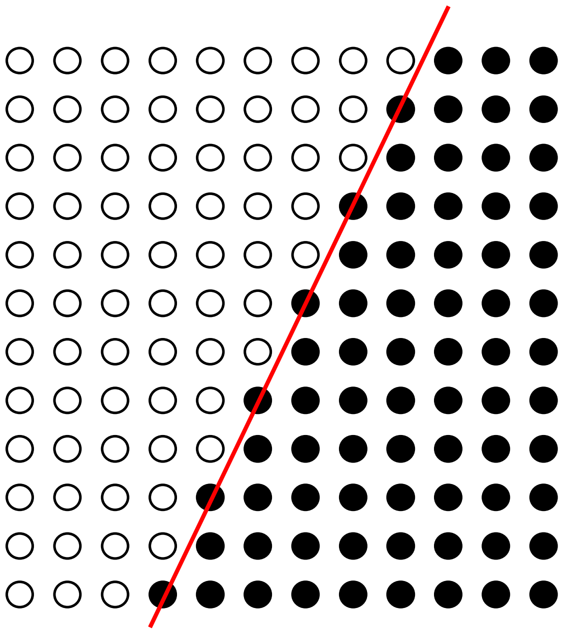
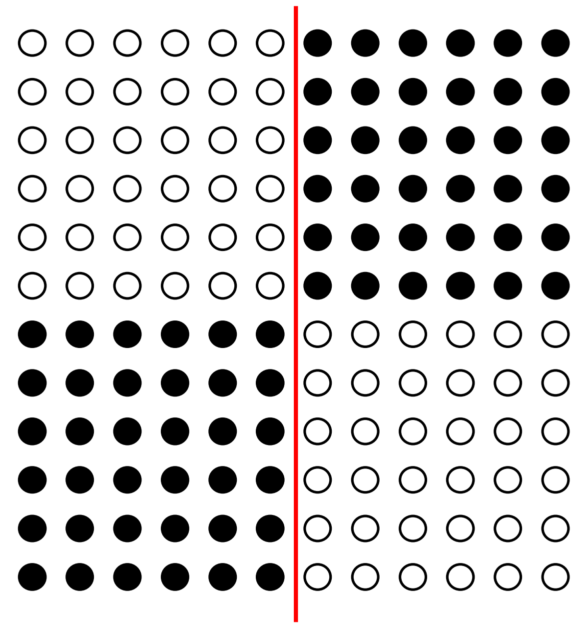


A half-plane



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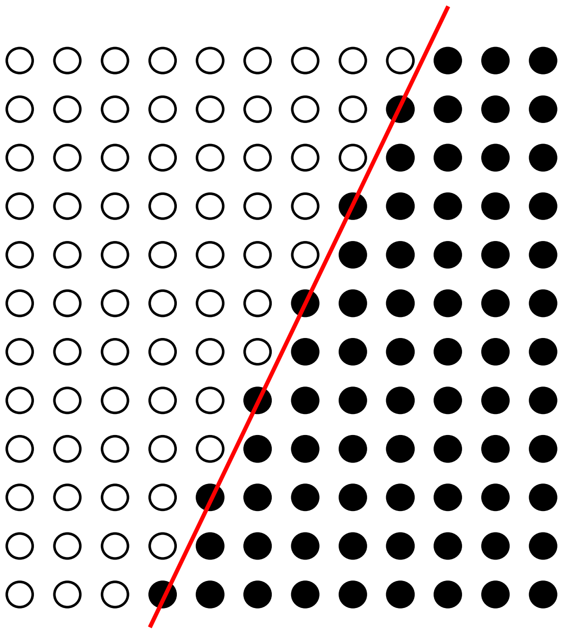
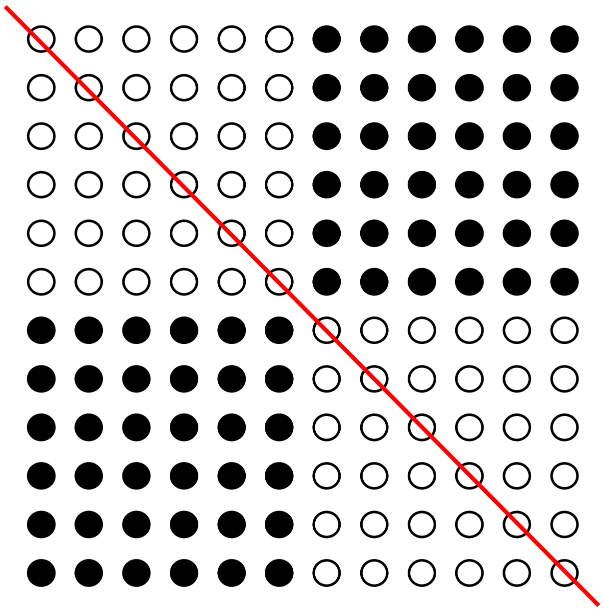
Half-plane Instances

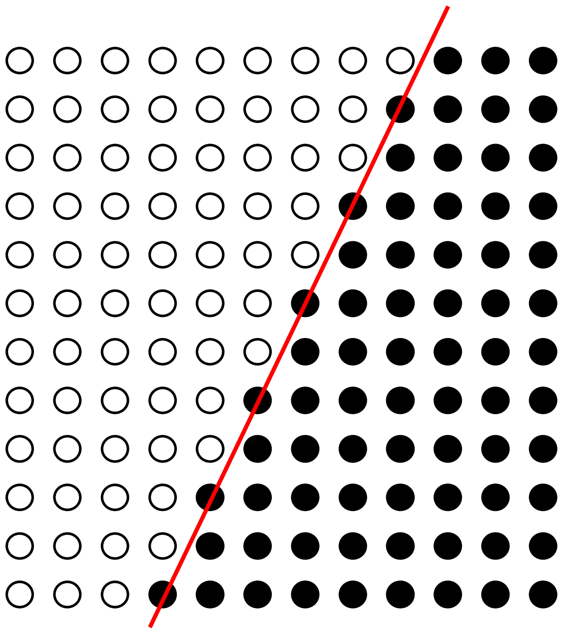


A half-plane

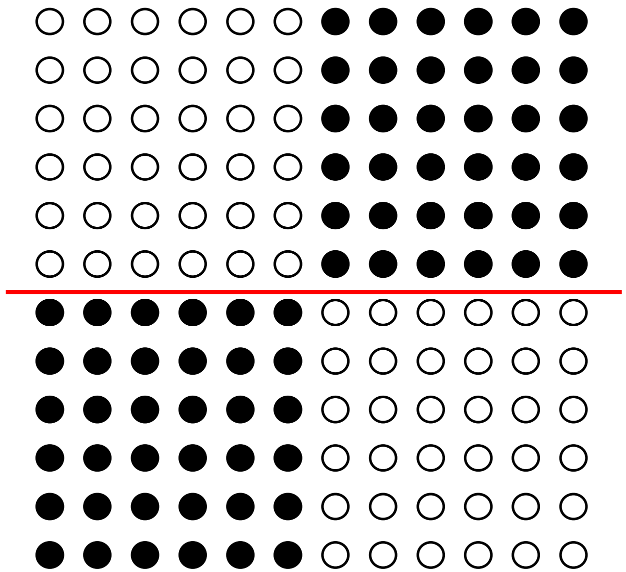


$\frac{1}{4}$-far from a half-plane

Half-plane Instances

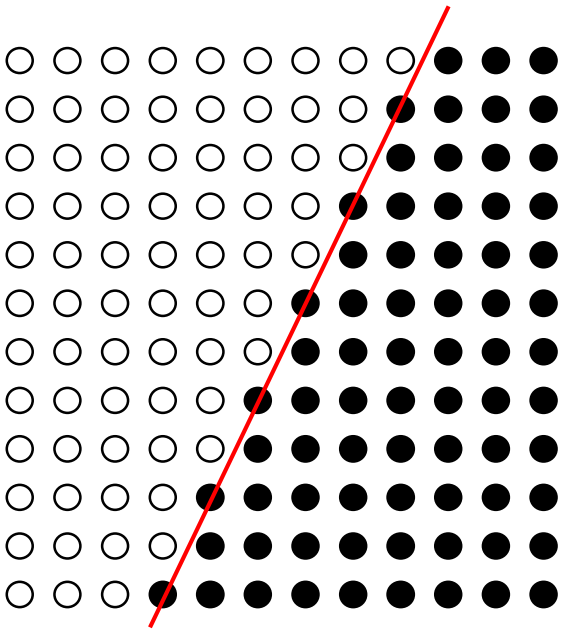


A half-plane

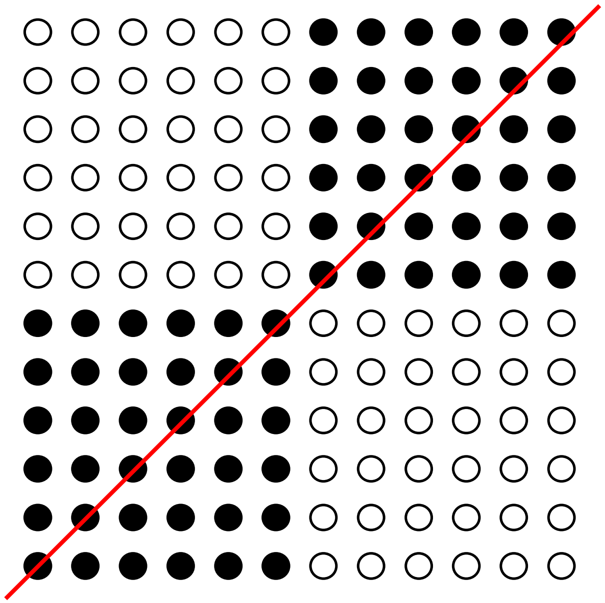


$\frac{1}{4}$-far from a half-plane

Half-plane Instances

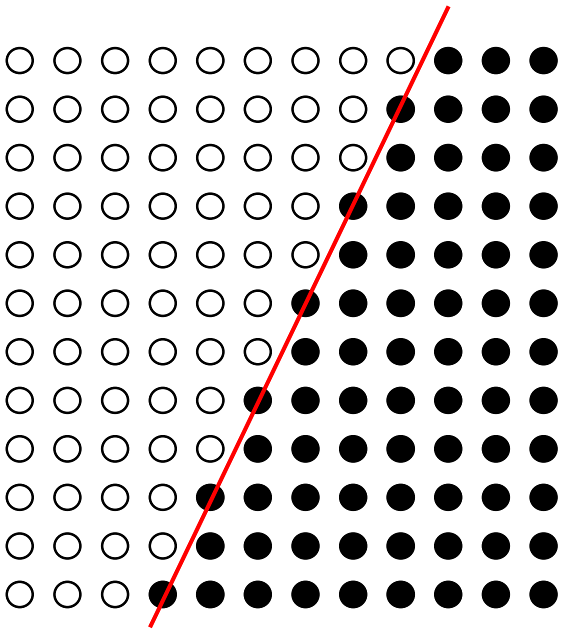


A half-plane

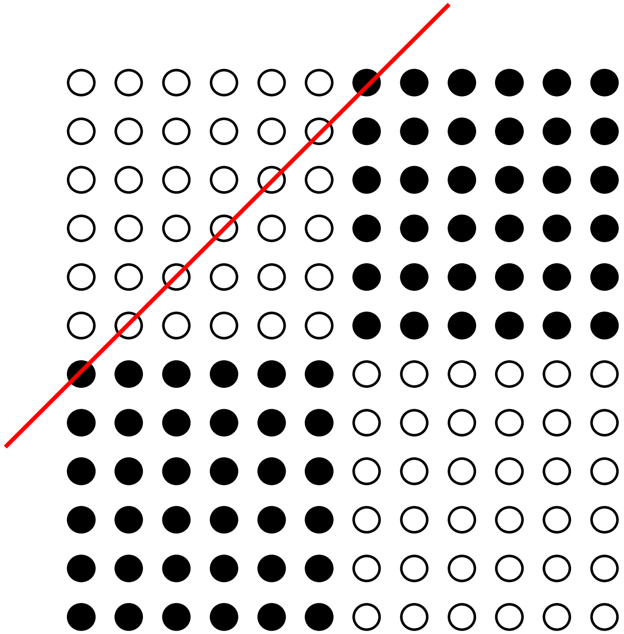


$\frac{1}{4}$-far from a half-plane

Half-plane Instances



A half-plane



$\frac{1}{4}$-far from a half-plane

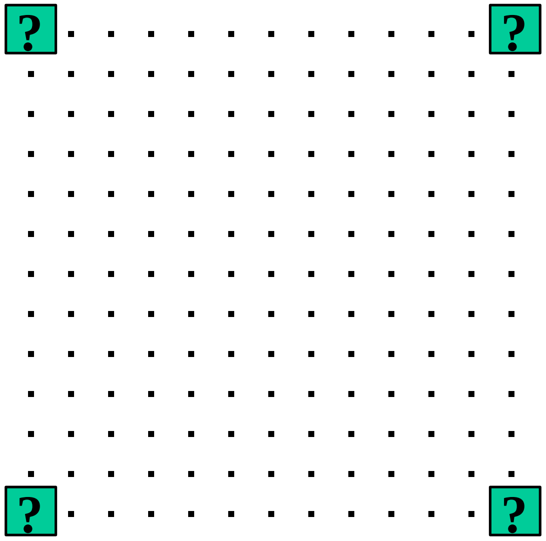
Strategy

"Testing by implicit learning" paradigm

- Learn the outline of the image by querying a few pixels.
- Test if the image conforms to the outline by random sampling, and reject if something is wrong.

Half-plane Test

Claim. The number of sides with different corners is 0, 2, or 4.

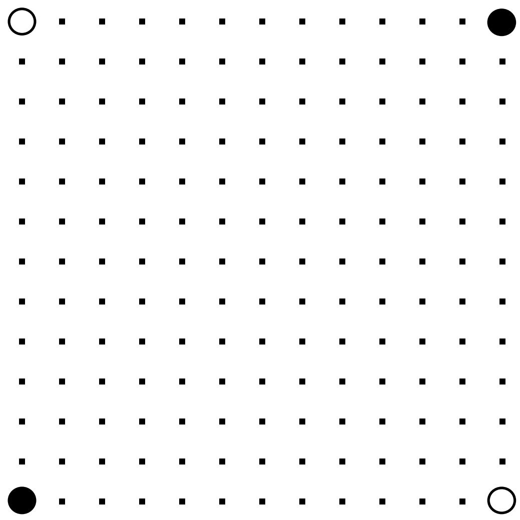


Algorithm

1. Query the corners.

Half-plane Test: 4 Bi-colored Sides

Claim. The number of sides with different corners is 0, 2, or 4.



Analysis

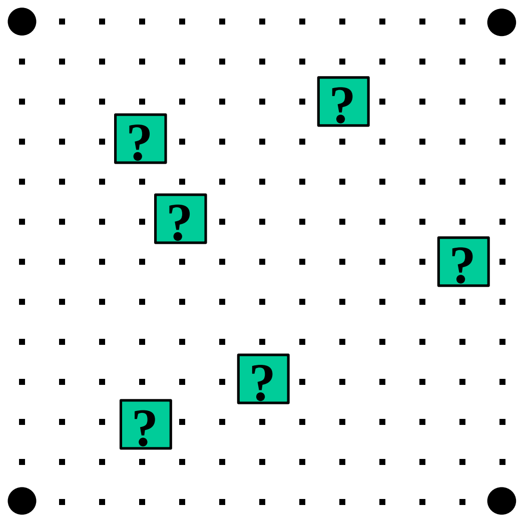
- If it is 4, the image cannot be a half-plane.

Algorithm

1. Query the corners.
2. If the number of sides with different corners is 4, **reject**.

Half-plane Test: 0 Bi-colored Sides

Claim. The number of sides with different corners is 0, 2, or 4.



Analysis

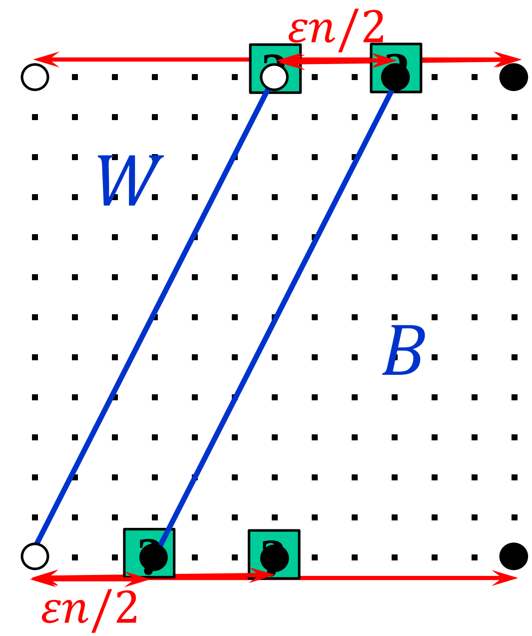
- If all corners have the same color, the image is a half-plane if and only if it is unicolored.

Algorithm

1. Query the corners.
2. If all corners have the same color c, test if all pixels have color c (as in Toy Example 1).

Half-plane Test: 2 Bi-colored Sides

Claim. The number of sides with different corners is 0, 2, or 4.



Analysis

- The area outside of $W \cup B$ has $\leq \varepsilon n^2/2$ pixels.
- If the image is a half-plane, W contains only white pixels and B contains only black pixels.
- If the image is ε-far from half-planes, it has $\geq \varepsilon n^2/2$ wrong pixels in $W \cup B$.
- By Witness Lemma, $4/\varepsilon$ samples suffice to catch a wrong pixel.

Algorithm

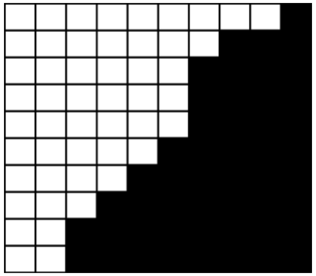
1. Query the corners.
2. If # of sides with different corners is 2, on both sides find 2 different pixels within distance $\varepsilon n/2$ by binary search.
3. Query $4/\varepsilon$ pixels from $W \cup B$
4. **Accept** iff all W pixels are white and all B pixels are black.

Testing if an Image is a Half-plane [R03]

A half-plane or

ε-far from a half-plane?

O(1/ε) time ✓



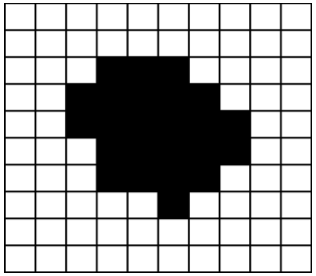
Other Results on Testing Properties of Images

- Pixel Model

  Convexity [Berman Murzabulatov R]

  Convex or ε-far from convex?

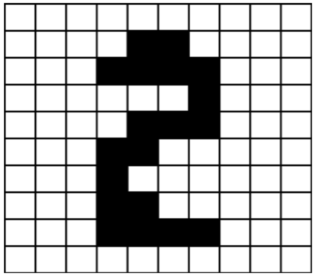
  $O(1/\varepsilon)$ time



  Connectedness [Berman Murzabulatov R]

  Connected or ε-far from connected?

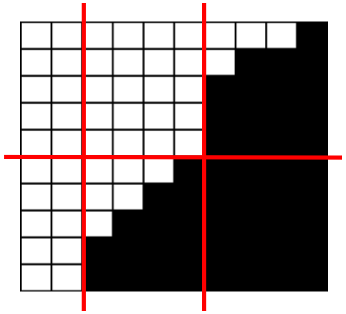
  $O(1/\varepsilon^{3/2}\sqrt{\log 1/\varepsilon}\ )$ time



  Partitioning [Kleiner Keren Newman 10]

  Can be partitioned according to a template or is ε-far?

  time independent of image size



- Properties of sparse images [Ron Tsur 10]