LECTURE 16

Last time

- Testing triangle-freeness
- Testing other properties of dense graphs
- Behrend's construction

Today

- Lower bound for testing triangle-freeness
- Canonical testers for the dense graph model

Project progress reports due today on Gradescope

Testing Triangle-Freeness

Input: parameters ε, n, access to undirected graph $G = (V, E)$ represented by $n \times n$ adjacency matrix.

Goal: **Accept** if G has no triangles;

reject w.p. $\geq \frac{2}{3}$ if G is ε-far from triangle-free

(at least $\varepsilon \binom{n}{2}$ edges need to be removed to get rid of all triangles).

- [Alon Fischer Krivelevich Szegedy 09]: Time that depends only on ε

- Today

Lower Bound for Testing Triangle-Freeness [Alon 02]

Testing triangle-freeness with 1-sided error requires super-polynomial dependence on $1/\varepsilon$.

$\Omega\left(\left(\frac{c}{\varepsilon}\right)^{c \log \frac{c}{\varepsilon}}\right)$ queries for some $c > 0$

Canonical Tester for Dense Graphs

Canonical Tester (**Input:** ε, n; query access to adjacency matrix of $G=(V,E)$)

1. Sample s nodes uniformly at random.
2. Query all pairs of sampled nodes.
3. **Accept** or **reject** based on available information.

- Consider any property $\mathcal{P}$ of graphs that does not depend on the names of the nodes. That is, if $G \in \mathcal{P}$ and G' is isomorphic to G then $G' \in \mathcal{P}$.

Exercise: Show that if there is an ε-tester T for $\mathcal{P}$ with query complexity $q(\varepsilon,n)$, then there is a canonical ε-tester T' for $\mathcal{P}$ with query complexity $O(q^2(\varepsilon,n))$. Moreover, if T has 1-sided error, so does T'.

A lower bound q for canonical tester implies a lower bound $\sqrt{q}$ for every tester

Sufficient to prove our lower bound $\Omega\left(\left(\frac{c}{\varepsilon}\right)^{c \log \frac{c}{\varepsilon}}\right)$ for 1-sided error canonical testers.

Exercise

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Goal for Proving the Lower bound

- A 1-sided error tester can reject only if it finds a triangle.
- Suppose we construct a graph G that is ε-far from being tringle free, where p fraction of triples are triangles for some small p.
- Consider a canonical tester T that samples q vertices.
- Let X be the number of triangles the tester catches.

$$\mathbb{E}[X] = p\binom{q}{3} = \Theta(p \cdot q^3)$$

- Suppose q is set so that $\mathbb{E}[X] \leq 1/2$
- By Markov, $\Pr[T \text{ rejects } G] \leq \Pr[X \geq 1] \leq \mathbb{E}[X] \leq \frac{1}{2} < \frac{2}{3}$
- So, for T to reject with high enough probability, $q = \Omega\left(p^{-\frac{1}{3}}\right)$

Sufficient to ensure $p = O\left(\left(\frac{\varepsilon}{c}\right)^{c \log \frac{c}{\varepsilon}}\right)$

Recall: Arithmetic-Progression-Free Sets

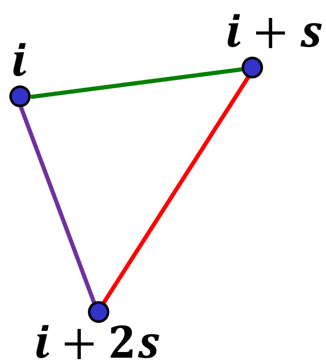
Behrend's Theorem

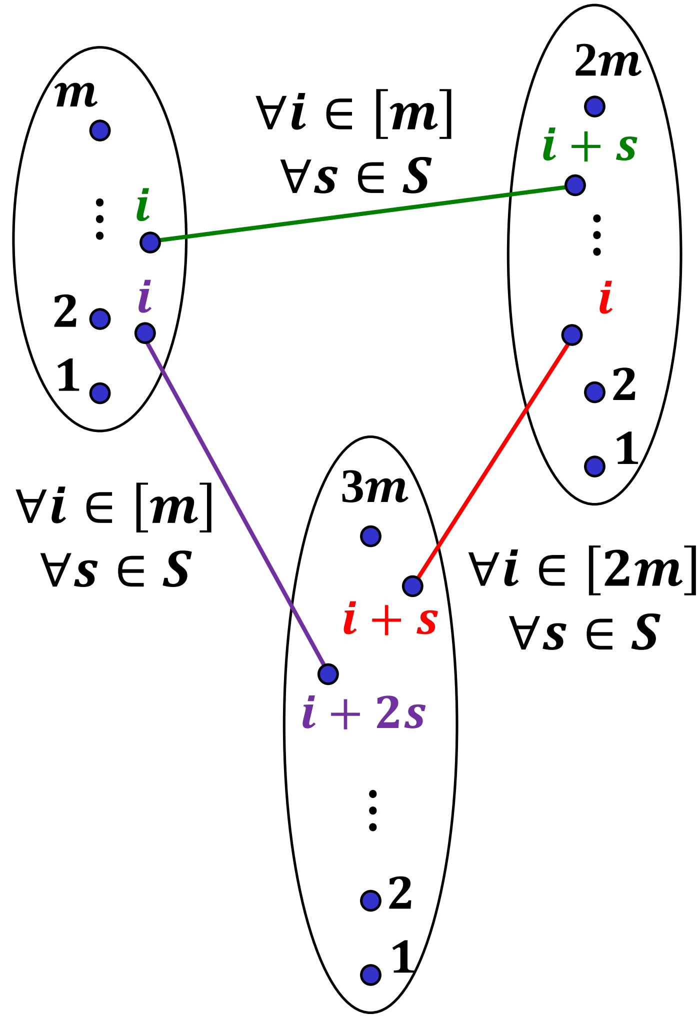
For all integer $m \geq 1$, there exists a set $S \subseteq [m]$ such that $|S| \geq \frac{m}{8^{\sqrt{\log_2 m}}}$ and the only solution to $x + y = 2z$ for $x, y, z \in S$ is $x = y = z$.

- We will use such a set S to construct a graph that is
  - far from triangle free
  - has relatively few triangles

Initial Graph Construction

- Let $S \subset [m]$ be a set from Behrend's Thm
- We construct a tripartite graph with
  m, $2m$, and $3m$ nodes in the three parts
- Intended triangles





- No other triangles:
  If $(i, i+x, i+x+y)$ is a triangle, then
  $x \in S, y \in S$, and $x + y = 2z$ for $z \in S$
  But then $x = y = z$ by construction of S
- All triangles are edge disjoint: each edge participates in exactly one triangle.

Parameters of the Initial Construction

- Number of nodes, n

  $6m$
- Number of edges

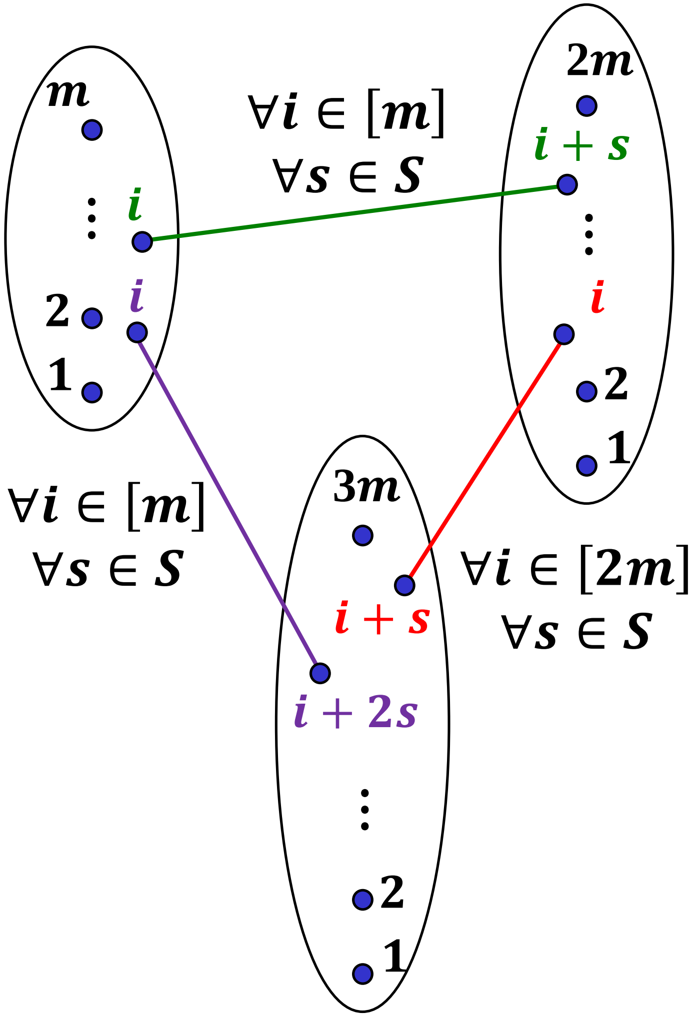
  $3m \cdot |S|$
- Number of (edge-disjoint) triangles, T

  $m \cdot |S|$
- Distance to triangle-freeness

Necessary and sufficient to remove one edge from each triangle, because they are edge-disjoint.

$$\frac{T}{\binom{n}{2}} = \Theta\left(\frac{m \cdot |S|}{m^2}\right) = \Theta\left(\frac{|S|}{m}\right) = \Theta\left(\frac{1}{8^{\sqrt{\log m}}}\right)$$

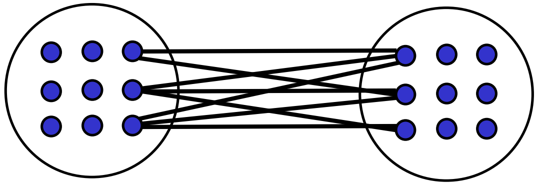
Not constant!



Blowup of a Graph

To construct a b-blowup of a graph,

- make b copies of each node;
- make two copies (of different nodes) adjacent iff their originals are adjacent.



Parameters of the Blowup Construction

- Number of nodes, n

  $6mb$
- Number of edges

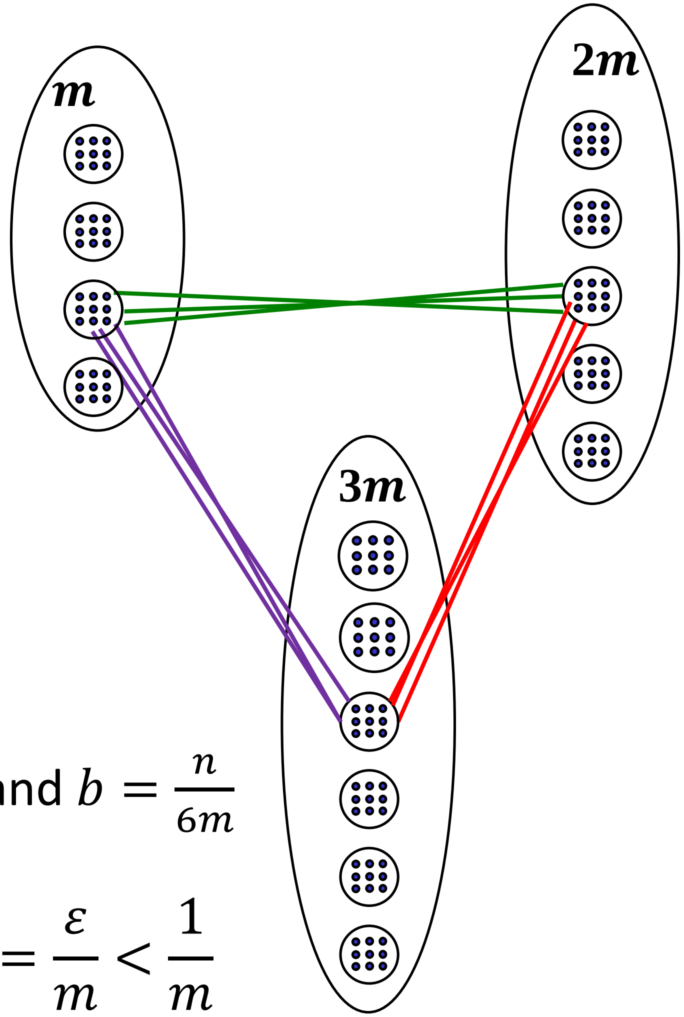
  $3m \cdot |S| \cdot b^2$
- Number of triangles

  $m \cdot |S| \cdot b^3$
- Number of (edge-disjoint) triangles, T

  $m \cdot |S| \cdot b^2$
- Distance to triangle-freeness

  $$\frac{T}{\binom{n}{2}} = \Theta\left(\frac{m \cdot |S| \cdot b^2}{(m \cdot b)^2}\right) = \Theta\left(\frac{|S|}{m}\right) = \Theta\left(\frac{1}{8\sqrt{\log m}}\right)$$
- Given ε and n, pick m so that $\varepsilon = \Theta\left(\frac{1}{8\sqrt{\log m}}\right)$ and $b = \frac{n}{6m}$
- Fraction of triples that are triangles:

  $$\approx \frac{m \cdot |S| \cdot b^3}{n^3} \approx \frac{m \cdot |S|}{m^3} = \frac{|S|}{m^2} = \frac{\varepsilon}{m} < \frac{1}{m}$$



Conclusion: Triangle-Freeness

- The query complexity of testing triangle-freeness with 1-sided error depends only on ε
  (and is independent of the size of the graph).

- However, the dependence is super-polynomial in $1/\varepsilon$