LECTURE 7

Last time

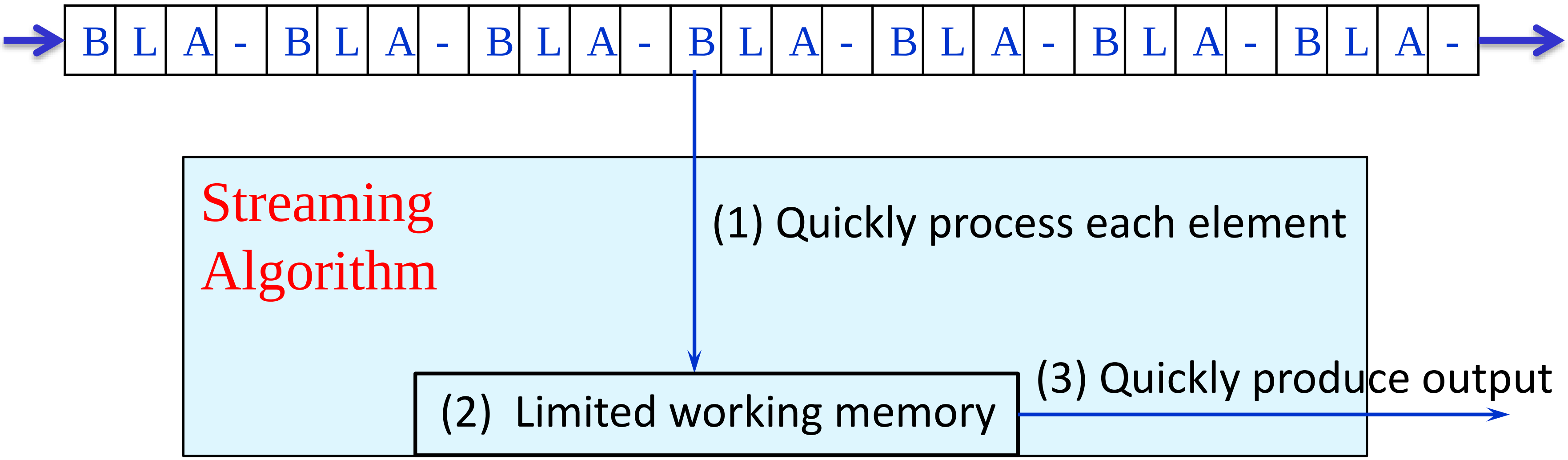
- Communication complexity
- Other models of computation

Today

- Streaming

Project proposals due next Thursday
Sign up for project meetings, scribing, grading

Data Stream Model [Alon Matias Szegedy 96]



Motivation: internet traffic analysis

Model the stream as m elements from $[n]$, e.g.,

$$\langle a_1, a_2, \ldots, a_m \rangle = 3, 5, 3, 7, 5, 4, \ldots$$

Goal: Compute a function of the stream, e.g., median, number of distinct elements, longest increasing sequence.

Based on Andrew McGregor's slides: http://www.cs.umass.edu/~mcgregor/slides/10-jhu1.pdf

Streaming Puzzle

A stream contains $n-1$ **distinct** elements from $[n]$ in arbitrary order.

Problem: Find the missing element, using $O(\log n)$ space.

Sampling from a Stream of Unknown Length

Warm-up: Find a uniform sample s from a stream $\langle a_1, a_2, \ldots, a_m \rangle$ of *known* length m.

Sampling from a Stream of Unknown Length

Problem: Find a uniform sample s from a stream $\langle a_1, a_2, \ldots, a_m \rangle$ of *unknown* length m

Algorithm (Reservoir Sampling)

1. Initially, $s \leftarrow a_1$
2. On seeing the t^{th} element, $s \leftarrow a_t$ with probability $1/t$

Analysis:

What is the probability that $s = a_i$ at some time $t \geq i$?

$$\Pr[s = a_i] = \frac{1}{i} \cdot \left(1 - \frac{1}{i+1}\right) \cdot \ldots \cdot \left(1 - \frac{1}{t}\right)$$
$$= \frac{1}{i} \cdot \frac{i}{i+1} \cdot \ldots \cdot \frac{t-1}{t} = \frac{1}{t}$$

Space: $O(k\ (\log n + \log m))$ bits to get k samples.

Counting Distinct Elements

Input: a stream $\langle a_1, a_2, \dots, a_m \rangle \in [n]^m$

Warm-up: Output the number of distinct elements in the stream.

Exact solutions:

- Store n bits, indicating whether each domain element has appeared.
- Store the stream: $O(m \log n)$ bits.

Known lower bounds:

- Every deterministic algorithm requires $\Omega(m)$ bits (even for a constant-factor approximation).
- Every exact algorithm (even randomized) requires $\Omega(n)$ bits.

Need to use both randomization and approximation to get polylog(m, n) space

Counting Distinct Elements

Input: a stream $\langle a_1, a_2, \ldots, a_m \rangle \in [n]^m$

Goal: Estimate the number of distinct elements in the stream up to a multiplicative factor $(1+\varepsilon)$ with probability $\geq 2/3$

- Studied by [Flajolet Martin 83, Alon Matias Szegedy 96,...]
- Today: $O(\varepsilon^{-2} \log n)$ space algorithm
  [Bar–Yossef Jayram Kumar Sivakuar Trevisan 02]
- Optimal: $O(\varepsilon^{-2} + \log n)$ space algorithm [Kane Nelson Woodruff 10]

Counting Distinct Elements

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Algorithm

1. Apply a random hash function $h : [n] \to [n]$ to each element
2. Compute X, the t-th smallest value of the hash seen where $t = 10 / \varepsilon^2$
3. Return $\tilde{r} = t \cdot n / X$ as estimate for r, the number of distinct elements.

Analysis:

- Algorithm uses $O(\varepsilon^{-2} \log n)$ bits of space (not accounting for storing h)
- We'll show: estimate $\tilde{r}$ has good accuracy with reasonable probability.

Claim. $\Pr[|\tilde{r} - r| \leq \varepsilon r] \geq 2/3$

Counting Distinct Elements: Analysis

Claim. $\Pr[|\tilde{r} - r| \le \varepsilon r] \ge 2/3$

X: t-th smallest hashed value

$t = 10 / \varepsilon^2$

$\tilde{r} = t \cdot n / X$

Proof: Suppose the distinct elements are $e_1, \dots, e_r$

- Overestimation:

$$\Pr[\tilde{r} \ge (1+\varepsilon) r] = \Pr\left[\frac{t \cdot n}{X} \ge (1+\varepsilon) r\right] = \Pr\left[X \le \frac{t \cdot n}{r(1+\varepsilon)}\right]$$

- Let $Y_i = \mathbb{1}\left[h(e_i) \le \frac{t \cdot n}{r(1+\varepsilon)}\right]$ and $Y = \sum_{i=1}^{r} Y_i$

$E[Y] = \frac{t}{1+\varepsilon}$

$\text{Var}[Y] \le E[Y]$

$$\mathrm{E}[Y] = r \cdot E[Y_1] = r \cdot \frac{t}{r(1+\varepsilon)} = \frac{t}{1+\varepsilon}$$

$$\mathrm{Var}[Y] = \mathrm{Var}\left[\sum_{i=1}^{r} Y_i\right] = \sum_{i=1}^{r} \mathrm{Var}[Y_i]$$

$$\le \sum_{i=1}^{r} \mathrm{E}\left[Y_i^2\right] = \sum_{i=1}^{r} \mathrm{E}[Y_i] = \mathrm{E}[Y]$$

Counting Distinct Elements: Analysis

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$E[Y] = \dfrac{t}{1+\varepsilon}$

$\mathrm{Var}[Y] \le E[Y]$

$$\Pr\left[X \le \frac{t \cdot n}{r(1+\varepsilon)}\right] = \Pr[Y \ge t] = \Pr[Y \ge (1+\varepsilon)\mathrm{E}[Y]]$$

- By the Chebyshev's inequality, for $\varepsilon \le 2/3$,

$$\Pr[Y \ge (1+\varepsilon)\mathrm{E}[Y]] \le \frac{\mathrm{Var}[Y]}{(\varepsilon \cdot \mathrm{E}[Y])^2} \le \frac{1}{\varepsilon^2 \mathrm{E}[Y]} = \frac{1+\varepsilon}{\varepsilon^2 \cdot t} = \frac{1+\varepsilon}{10} \le \frac{1}{6}$$

- Underestimation: A similar analysis shows $\Pr[\tilde{r} \le (1-\varepsilon) r] \le \frac{1}{6}$

Removing the Random Hashing Assumption

Idea: Use limited independence

- A family $\mathcal{H} = \{h: [a] \to [b]\}$ of hash functions is k-wise independent if for all distinct $x_1, \ldots, x_k \in [a]$ and all $y_1, \ldots, y_k \in [b]$,

$$\Pr_{h \in \mathcal{H}}[h(x_1) = y_1, \ldots, h(x_k) = y_k] = \frac{1}{b^k}$$

  Note: a uniformly random family is k-wise independent for all k

- Observations: For $x_1, \ldots, x_k$ as above,
  1. $h(x_1)$ is uniform over $[b]$
  2. $h(x_1), \ldots, h(x_k)$ are mutually independent.

Construction of k-wise Independent Family

Idea: Use limited independence

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$$\Pr_{h \in \mathcal{H}}[h(x_1) = y_1, \ldots, h(x_k) = y_k] = \frac{1}{b^k}$$

Construction of k-wise Independent Family of Hash Functions

1. Let p be a prime.
2. Condider the set of polynomials of degree $k-1$ over $\mathbb{F}_p$

$$\mathcal{H} = \{h: \{0, \ldots, p-1\} \to \{0, \ldots, p-1\} \mid h(x) = c_{k-1}x^{k-1} + \cdots + c_1 x + c_0, \text{with } c_0, \ldots, c_{k-1} \in \mathbb{F}_p\}$$

3. To sample $h \in \mathcal{H}$, sample $c_0, \ldots, c_{k-1} \in \mathbb{F}_p$ u.i.r.

- Space to store h is $O(k \log p)$
- For arbitrary a, b, need $O\big(k \cdot (\log a + \log b)\big)$ space.

Counting Distinct Elements: Final Algorithm

Input: a stream $\langle a_1, a_2, \dots, a_m \rangle \in [n]^m$

Goal: Estimate the number of distinct elements in the stream up to a multiplicative factor $(1 + \varepsilon)$ with probability $\geq 2/3$

Algorithm

1. Sample a hash function $h : [n] \to [n]$ from a 2-wise independent family and apply h to each element
2. Compute X, the t-th smallest value of the hash seen where $t = 10 \,/\, \varepsilon^2$
3. Return $\tilde{r} = t \cdot n / X$ as estimate for r, the number of distinct elements.

Analysis:

- Algorithm uses $O(\varepsilon^{-2} \log n)$ bits of space
- Our correctness analysis applies.

Frequency Moments Estimation

Input: a stream $\langle a_1, a_2, \ldots, a_m \rangle \in [n]^m$

- The frequency vector of the stream is $f = (f_1, \ldots, f_m)$,
  where f_i is the number of times i appears in the stream
- The p-th frequency moment is $F_p = \left|\left|f\right|\right|_p^p = \sum_{i=1}^n f_i^p$

  F_0 is the number of nonzero entries of f (# of distinct elements)

  $F_1 = m$ (# of elements in the stream)

  $F_2 = \left|\left|f\right|\right|_2^2$ is a measure of non-uniformity

  used e.g. for anomaly detection in network analysis

  $F_\infty = \max_i f_i$ is the most frequent element

Goal: Estimate F_p up to a multiplicative factor $(1 + \varepsilon)$ with probability $\geq 2/3$

Summary

Streaming Model

- Reservoir sampling
- Distinct Elements (approximating F_0)
- k-wise independent hashing