Homework 3 – Due Thursday, April 5 *before* 10am on Angel

Reminder: Final project reports are due Tuesday, April 17.

Instructions

- Solutions written in LaTeX are strongly preferred, but you can upload any pdf files, including scanned hand-written solutions. Template latex files are on the course webpage.
- Collaboration is allowed and encouraged. However, each of you should think about a problem before discussing it with others and write up your solution independently. You may consult books and on-line sources to get information about well-known theorems, such as the Chernoff bound. **But you are not allowed to look up solutions directly in papers or any other sources.** And you *must* list all collaborators and sources!
- Correctness, clarity, and succinctness of the solution will determine your score.

Problems

1. [**Application of the Regularity Lemma**] In this problem, you are asked to show that the number of triangle-free labeled graphs on n nodes is $2^{\left(\frac{1}{4}+o(1)\right)n^2}$.

   (a) Show that there are at least $2^{\frac{1}{4}n^2}$ triangle-free labeled graphs on n nodes.

   (b) Prove (e.g., by induction) that every triangle-free graph contains at most $\frac{n^2}{4}$ edges.

   (c) Using the Regularity Lemma and 1b, show that there are at most $2^{\left(\frac{1}{4}+o(1)\right)n^2}$ triangle-free labeled graphs on n nodes.

   You may use the following facts without a proof:

$$\binom{n}{\alpha n} \leq 2^{n(H(\alpha)+o(1))} \text{ where } H(\alpha) = -\alpha\log\alpha - (1-\alpha)\log(1-\alpha). \tag{1}$$

$$\text{The entropy function } H(\alpha) \text{ tends to 0 as } \alpha \text{ tends to 0.} \tag{2}$$

2. [**A lower bound for testing monotonicity on a hypercube**] In this problem you will prove that every non-adaptive 1-sided error ϵ-tester for monotonicity of functions $f : \{0,1\}^d \to \{0,1\}$ makes $\Omega(\sqrt{d})$ queries and deduce a lower bound for 1-sided error adaptive ϵ-testers. (Make sure you understand that a lower bound on query complexity implies the same lower bound on the running time.)

   (a) For $x \in \{0,1\}^d$, let $\|x\|$ denote the weight of x, i.e., the number of 1s in vector x. For $i = 1, ..., d$ define a function $f_i : \{0,1\}^d \to \{0,1\}$ by

$$f_i(x_1 \dots x_d) \quad = \quad \begin{cases} 1 & \text{if } \|x\| > d/2 + \sqrt{d}; \\ 0 & \text{if } \|x\| < d/2 - \sqrt{d}; \\ 1 - x_i & \text{otherwise.} \end{cases}$$

Prove that for all $1 \leq i \leq d$, f_i is ϵ-far from monotone, for some constant $\epsilon > 0$ independent of d.

(b) Define a distribution $\mathcal{D}$ on the inputs that you will use to apply Yao's principle.

(c) Define a *violation* (of monotonicity) and give an upper bound on the probability that any fixed 2 queries will detect a violation under $\mathcal{D}$.

(d) Do the same for q queries.

(e) What is the best lower bound on 1-sided error adaptive tests you can deduce?

(f) (Optional, hard) Prove a lower bound that applies to 2-sided error tests.

3. **[Testing properties defined by monotone kCNFs]** A *monotone* CNF formula is an AND of OR-clauses (no negations); in a monotone kCNF each OR-clause has at most k variables. Design an ϵ-tester that, given an input assignment $x_1, \ldots, x_n \in \{0,1\}$ to a fixed in advance monotone kCNF formula on n variables, accepts if the input satisfies the formula and rejects with probability at least $\frac{2}{3}$ if the input is ϵ-far, i.e., at least ϵn bits have to be flipped to make it satisfy the formula. The *query* complexity of the tester has to be $O(n^{\frac{k-1}{k}})$, but there are no restrictions on the running time.

   Hints: 1. Notice that the string of all 1s satisfies a monotone formula. Prove that if a string is ϵ-far from satisfying a monotone formula, at least $\epsilon n/k$ variable disjoint clauses of the formula are falsified. 2. Modify the tester for monotonicity on general graphs.