LECTURE 13

Last time

- Graph property testing (for dense graphs)
- Testing bipartiteness

Today

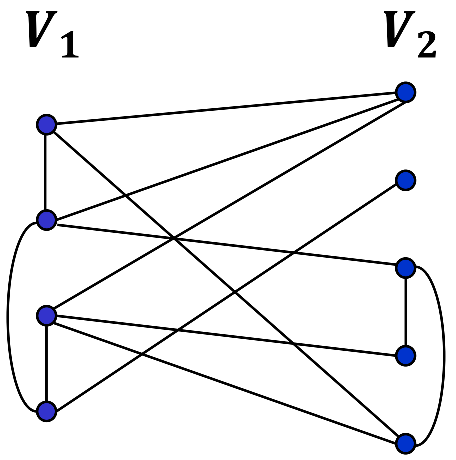
Approximate Max-Cut

[Goldreich Goldwasser Ron 98]

Max Cut in Dense Graphs

- Let $G = (V, E)$ be an undirected n-node graph.

- Let (V_1, V_2) be a partition of V.
  $e(V_1, V_2)$ = set of edges crossing the cut



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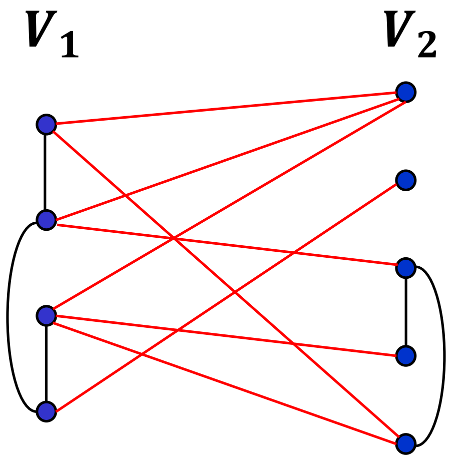


- The edge density of the cut,
  denoted $\mu(V_1, V_2)$, is $\frac{|e(V_1,V_2)|}{n^2}$.

- The edge density of the largest cut in G is
$$\mu(G) = \max_{(V_1,V_2)} \mu(V_1, V_2)$$

Approximate Max-Cut Problem

Input: parameter ε, access to undirected graph $G = (V, E)$ represented by $n \times n$ adjacency matrix.

Goal 1: Output an estimate $\hat{\mu}$ such that:

$$\Pr[|\hat{\mu} - \mu(G)| \leq \varepsilon] \geq 2/3$$

- [GGR98]: poly$\left(\frac{1}{\varepsilon}\right)$ queries and $O(2^{poly\left(\frac{1}{\varepsilon}\right)})$ time

Goal 2: Output a partition (V_1, V_2) with edge density

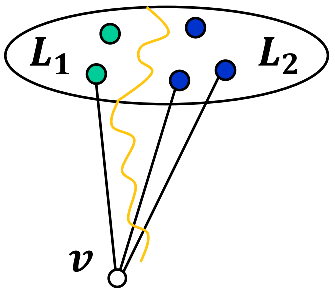
$$\mu(V_1, V_2) \geq \mu(G) - \varepsilon$$

with probability at least 2/3.

- [GGR98]: $O\left(2^{poly\left(\frac{1}{\varepsilon}\right)} + n \cdot poly\left(\frac{1}{\varepsilon}\right)\right)$ time

Greedy Partitioning

- Suppose we have a partition (L_1, L_2) of $L \subset V$.
- In which part should we place a new node v to maximize edge density?



- Let $\Gamma(v, U)$ be the number of neighbors of v in U.

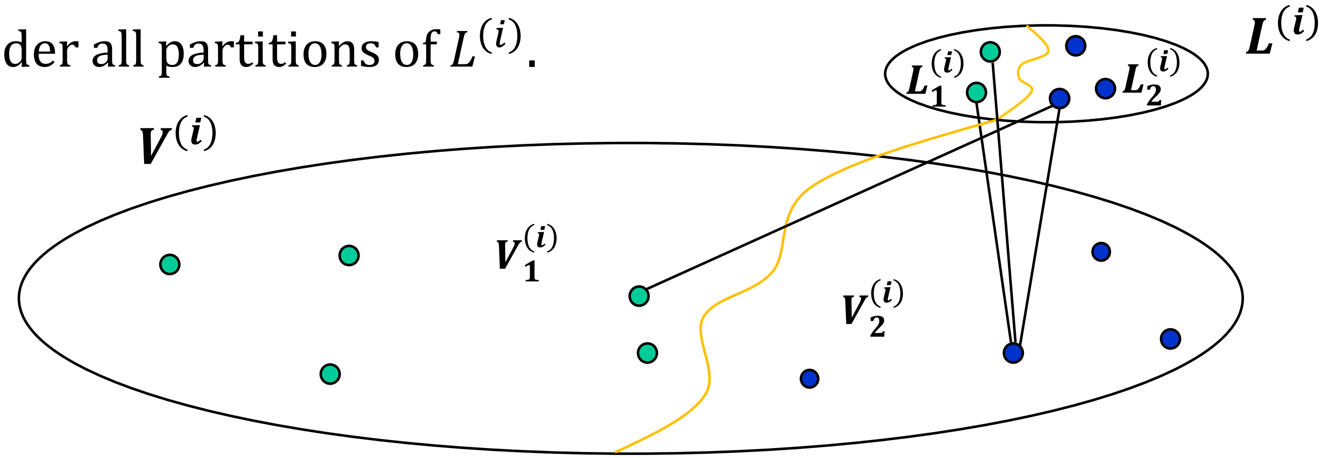
- Greedy: If $\Gamma(v, L_1) \leq \Gamma(v, L_2)$, place v in L_1; otherwise, place it in L_2.

Main Idea

- Partition V into sets $V^{(i)}$ of (almost) equal size.

Assume they are of equal size.

- For each set $V^{(i)}$, sample a learning set $L^{(i)}$ from the vertices not in $V^{(i)}$.
- Consider all partitions of $L^{(i)}$.



A partition of $L^{(i)}$ induces a partition of $V^{(i)}$ via the greedy rule

A partition sequence $\pi(L) = \left(\left(L_1^{(1)}, L_2^{(1)}\right), \ldots, \left(L_1^{(t)}, L_2^{(t)}\right)\right)$ induces a partition of V

- Consider all such partitions of V and pick the best.

Preliminary Max-Cut Approximation Algorithm

Algorithm (**Input:** ε, n; query access to adjacency matrix of $G=(V,E)$)

1. Partition V into $t = 4/\varepsilon$ sets $V^{(1)}, V^{(2)}, \dots, V^{(t)}$ of (almost) equal size.
2. For each $i \in [t]$, selelect a set $L^{(i)}$ of size $\ell = \frac{1}{\varepsilon^2} \cdot \log\frac{1}{\varepsilon}$ u.i.r. from $V \backslash V^{(i)}$.
   Let $L = \left(L^{(1)}, L^{(2)}, \dots, L^{(t)}\right)$.
3. For each partition sequence $\pi(L) = \left(\left(L_1^{(1)}, L_2^{(1)}\right), \dots, \left(L_1^{(t)}, L_2^{(t)}\right)\right)$
4. For each $i \in [t]$
5. Partition $V^{(i)}$ into $\left(V_1^{(i)}, V_2^{(i)}\right)$ using the greedy rule:
   place v in $V_1^{(i)}$ iff $\Gamma(v, L_1) \le \Gamma(v, L_2)$.
6. Let $V_1^{\pi} = \cup_i V_1^{(i)}$ and $V_2^{\pi} = \cup_i V_2^{(i)}$; calculate $\mu(V_1^{\pi}, V_2^{\pi})$.
7. Output the cut (V_1^{π}, V_2^{π}) with the largest density.

- Number of partition sequences: $\left(2^{\ell}\right)^t = 2^{poly\left(\frac{1}{\varepsilon}\right)}$
- Running time: $n^2 \cdot 2^{poly\left(\frac{1}{\varepsilon}\right)}$

$O\left(n^2\right)$ time for calculating each density

Correctness of Max-Cut Approximation

Correctness Theorem

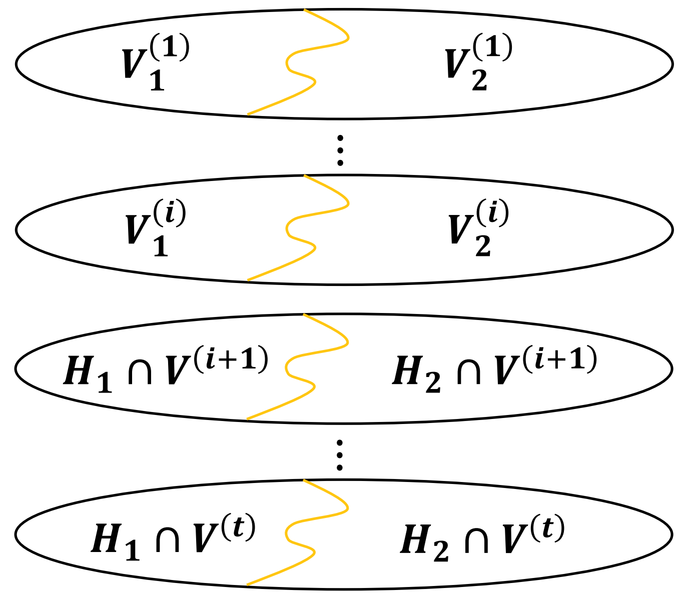
Let (H_1, H_2) be a partition of V.

w. p. $\geq 5/6$ over the choice of L, some partition sequence $\pi(L)$ induces a partition (V_1^π, V_2^π) of V s. t. $\mu(V_1^\pi, V_2^\pi) \geq \mu(H_1, H_2) - 3\varepsilon/4$

Think: (H_1, H_2) is a max-cut

Main Proof Idea: Use a hybrid argument.

- $\left(H_1^{(0)}, H_2^{(0)}\right) = (H_1, H_2)$
- For $i \in [t]$, partition $\left(H_1^{(i)}, H_2^{(i)}\right)$ is obtained from $\left(H_1^{(i-1)}, H_2^{(i-1)}\right)$ by repartitioning $V^{(i)}$ into into $\left(V_1^{(i)}, V_2^{(i)}\right)$, the best out of all partitions induced by a partition of $L^{(i)}$.
- We will show that when we go from one hybrid to the next, the density does not drop too much (w.h.p.)



i-th hybrid partition $\left(H_1^{(i)}, H_2^{(i)}\right)$

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Proof: Consider $i \in [t]$ and fix learning sets $L^{(1)}, \ldots, L^{(i-1)}$.

- Let A_i be the event that $\mu\left(H_1^{(i)}, H_2^{(i)}\right) \geq \mu\left(H_1^{(i-1)}, H_2^{(i-1)}\right) - \frac{3\varepsilon}{4t}$

Main Lemma

$\Pr[A_i] \geq 1 - \frac{1}{6t}$, where the probability is taken over the choice of $L^{(i)}$.

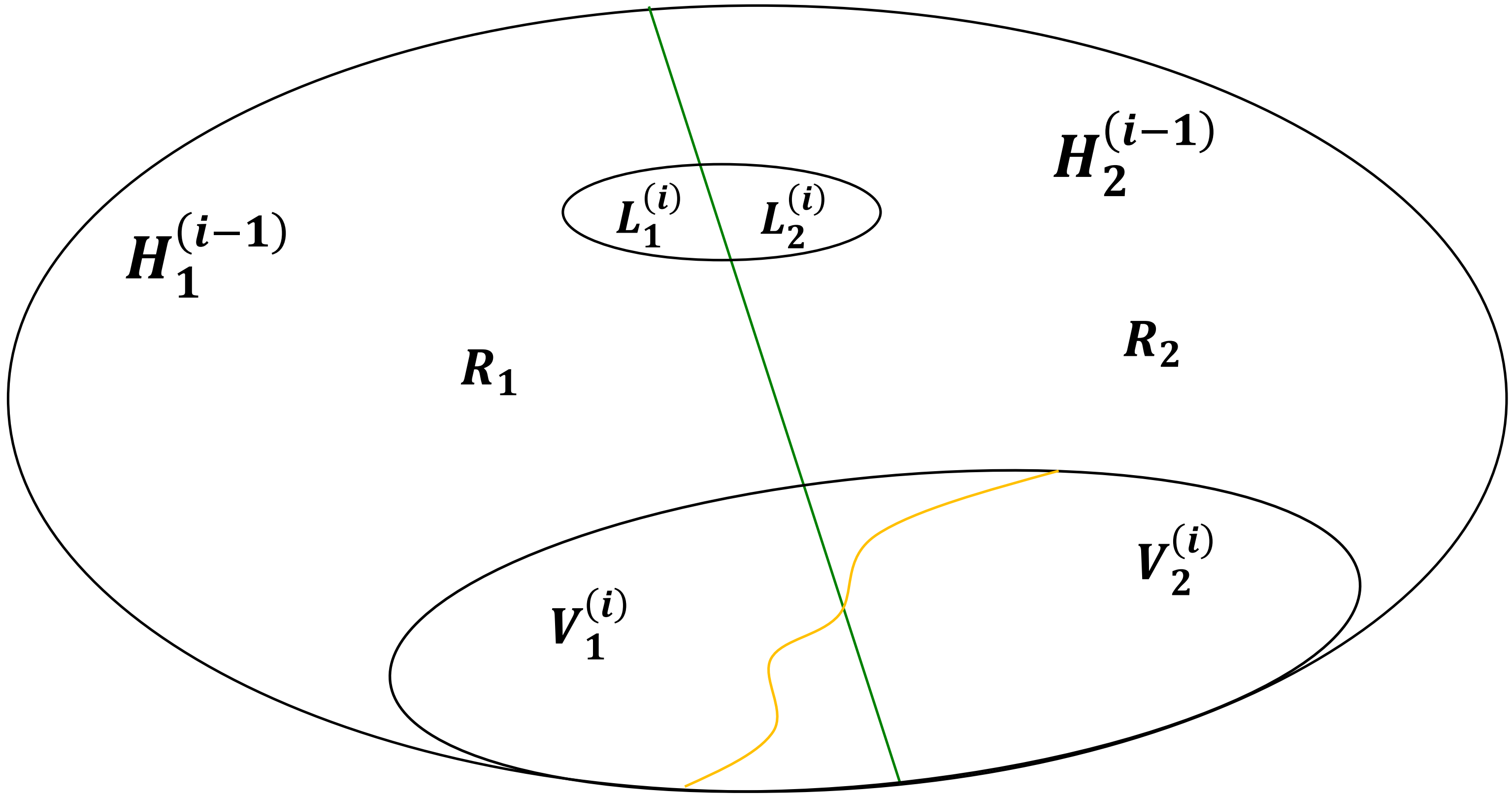
- Then, by a union bound,

$$\Pr\left[\bigcup_i \overline{A_i}\right] \leq t \cdot \frac{1}{6t} = \frac{1}{6}$$

Big Picture

When we go from hybrid $i-1$ to hybrid i, only nodes in $V^{(i)}$ get repartitioned.

- Let $R_1 = V \backslash \mathrm{V}^{(i)} \cap H_1^{(i-1)}$ and $R_2 = V \backslash \mathrm{V}^{(i)} \cap H_2^{(i-1)}$
- Let $L_1^{(i)} = L^{(i)} \cap H_1^{(i-1)}$ and $L_2^{(i)} = L^{(i)} \cap H_2^{(i-1)}$



Proof of Main Lemma

- A node v is good w.r.t. $\left(L_1^{(i)}, L_2^{(i)}\right)$ if $\left|\frac{\Gamma\left(v, L_j^{(i)}\right)}{\ell} - \frac{\Gamma(v, R_j)}{n}\right| \leq \frac{\varepsilon}{8}\ \forall j \in \{1,2\}$
- Learning set $L^{(i)}$ is good for (R_1, R_2) if $\leq \frac{\varepsilon}{4}$ fraction of nodes in $V^{(i)}$ are bad

Claim 1

The probability that all t learning sets are good is at least 5/6.

- A node v is balanced w.r.t. (R_1, R_2) if $\left|\frac{\Gamma(v, R_1)}{n} - \frac{\Gamma(v, R_2)}{n}\right| \leq \frac{\varepsilon}{4}$

Observation

If all learning sets are good, all good unbalanced nodes are placed correctly.

Proof: Suppose w.l.o.g. that $\Gamma(v, R_1) \leq \Gamma(v, R_2)$ for a good unbalanced node v

$$\frac{\varepsilon}{4} < \frac{\Gamma(v, R_2)}{n} - \frac{\Gamma(v, R_1)}{n} \leq \left(\frac{\Gamma\left(v, L_2^{(i)}\right)}{n} + \frac{\varepsilon}{8}\right) - \left(\frac{\Gamma\left(v, L_1^{(i)}\right)}{n} - \frac{\varepsilon}{8}\right)$$

So, $\Gamma\left(v, L_1^{(i)}\right) < \Gamma\left(v, L_2^{(i)}\right)$, and v is placed correctly.

Density Loss from Repartitioning $V^{(i)}$

when $\left(L_1^{(i)}, L_2^{(i)}\right)$ is good

Type of cut-edges	Number of edges lost
Incident to good unbalanced nodes	
Incident to bad unbalanced nodes	
Incident to balanced nodes	
Between nodes of $V^{(i)}$	

Total: $\frac{3\varepsilon}{4t} \cdot n^2$

- Recall: A_i is the event that $\mu\left(H_1^{(i)}, H_2^{(i)}\right) \geq \mu\left(H_1^{(i-1)}, H_2^{(i-1)}\right) - \frac{3\varepsilon}{4t}$
- When $\left(L_1^{(i)}, L_2^{(i)}\right)$ is good, A_i occurs.
- It remains to show that w.p. $\geq 5/6$ all learning sets are good.

Probability of Good Learning Sets

- A node v is good w.r.t. $\left(L_1^{(i)}, L_2^{(i)}\right)$ if $\left|\frac{\Gamma\left(v,\, L_j^{(i)}\right)}{\ell} - \frac{\Gamma(v,\, R_j)}{n}\right| \le \frac{\varepsilon}{8}\ \forall j \in \{1,2\}$
- Learning set $L^{(i)}$ is good for (R_1, R_2) if $\le \frac{\varepsilon}{4}$ fraction of nodes in $V^{(i)}$ are bad

Claim 1

The probability that all t learning sets are good is at least 5/6.

Proof: It suffices to prove that Pr[$L^{(i)}$ is bad]$\le \frac{1}{6t}$

- Fix $v \in V^{(i)}$
- Let $L^{(i)} = \{v_1, \dots, v_\ell\}$. Recall that it is chosen u.i.r. from $V \setminus \mathrm{V}^{(i)}$

$$X_j^k = \begin{cases} 1, & \text{if } v_k \text{ is a neibhbor of } v \text{ in } R_j \\ 0, & \text{otherwise} \end{cases} \quad \forall j \in \{1,2\}$$

$$X_j = \sum_{k \in [\ell]} X_j^k = \Gamma\left(v, L_j^{(i)}\right)$$

$$\mathbb{E}[X_j] = \sum_{k \in [\ell]} \mathbb{E}[X_j^k] = \ell \cdot \frac{1}{n} \Gamma(v, R_j)$$

- Use Hoeffding Bound.

Improved Max-Cut Approximation Algorithm

Algorithm (**Input:** ε, n; query access to adjacency matrix of $G=(V,E)$)

1. Partition V into $t = 4/\varepsilon$ sets $V^{(1)}, V^{(2)}, \dots, V^{(t)}$ of (almost) equal size.
2. For each $i \in [t]$, selelect a set $L^{(i)}$ of size $\ell = \frac{1}{\varepsilon^2} \cdot \log\frac{1}{\varepsilon}$ u.i.r. from $V \setminus V^{(i)}$. Let $L = \left(L^{(1)}, L^{(2)}, \dots, L^{(t)}\right)$.
3. Select u.i.r. S of size $m = \frac{t\ell}{\varepsilon^2}$
4. For each partition sequence $\pi(L) = \left(\left(L_1^{(1)}, L_2^{(1)}\right), \dots, \left(L_1^{(t)}, L_2^{(t)}\right)\right)$
5. For each $i \in [t]$
6. Partition $S^{(i)}$ into $\left(S_1^{(i)}, S_2^{(i)}\right)$ using the greedy rule: add v to $S_1^{(i)}$ iff $\Gamma(v, L_1) \le \Gamma(v, L_2)$.
7. Let $S_1^{\pi} = \cup_i S_1^{(i)}$ and $S_2^{\pi} = \cup_i S_2^{(i)}$; calculate

$$\mu'(S_1^{\pi}, S_2^{\pi}) = \frac{|\{k : \{s_{2k-1}, s_{2k}\} \in e(S_1^{\pi}, S_2^{\pi})\}|}{m/2}$$

8. Output $\max_{\pi} \mu'(S_1^{\pi}, S_2^{\pi})$

- We can also out put the cut of V induced by π with max μ'