Sublinear Algorithms

LECTURE 24

Last time

- L_p-testing of monotonicity
- Work investment strategy
- Testing via learning

- **Today**
- Finish testing via learning
- Local Computation Algorithms (LCAs)
- Distributed LOCAL model
- Maximal Independent Set (MIS)

Project Reports are due Thursday, presentations next week

Monotonicity Testers: Running Time

f	L_0	L_p
$[n] \to \{0,1\}$	$\Theta\left(\frac{1}{\boldsymbol{\varepsilon}}\right)$	$\Theta\left(\frac{1}{\boldsymbol{\varepsilon}^p}\right)$
$[n]^d \to \{0,1\}$	$O\left(\frac{d}{\boldsymbol{\varepsilon}} \cdot \log \frac{d}{\boldsymbol{\varepsilon}}\right)$	$O\left(\frac{d}{\boldsymbol{\varepsilon}^p} \log \frac{d}{\boldsymbol{\varepsilon}^p}\right)$ $\Omega\left(\frac{1}{\boldsymbol{\varepsilon}^p} \log \frac{1}{\boldsymbol{\varepsilon}^p}\right)$ for $d = 2$ nonadaptive 1-sided error
		$\Theta\left(\frac{1}{\boldsymbol{\varepsilon}^p}\right)$ for constant d adaptive 1-sided error

Testing Monotonicity of $\boldsymbol{f}: [\boldsymbol{n}]^{\mathbf{2}} \to \{\mathbf{0}, \mathbf{1}\}$

- For nonadaptive, 1-sided error testers, $\Omega\left(\frac{1}{\varepsilon}\log\frac{1}{\varepsilon}\right)$ queries are needed.
- There is an adaptive, 1-sided error tester with $O\left(\frac{1}{\varepsilon}\right)$ queries.
  Method: testing via learning.

Partial Learning

- An ε-partial function g with domain D and range R is a function $g : D \to R \cup \{?\}$ that satisfies $\Pr_{x \in D}[g(x) = ?] \le \varepsilon$.
- An ε-partial function g agrees with a function f if $g(x) = f(x)$ for all x on which $g(x) \neq ?$.
- Given a function class $\boldsymbol{\mathcal{C}}$, let $\boldsymbol{\mathcal{C}_\varepsilon}$ denote the class of ε-partial functions, each of which agrees with some function in $\boldsymbol{\mathcal{C}}$.
- An ε-partial learner for a function class $\boldsymbol{\mathcal{C}}$ is an algorithm that, given a parameter ε and oracle access to a function f, outputs a hypothesis $g \in \boldsymbol{\mathcal{C}_\varepsilon}$ or fails. Moreover, if $f \in \boldsymbol{\mathcal{C}}$ then it outputs g that agrees with f.

Lemma (Conversion from Learner to Tester)

If there is an ε-partial learner for a function class $\boldsymbol{\mathcal{C}}$ that makes $q(\varepsilon)$ queries then $\boldsymbol{\mathcal{C}}$ can be ε-tested with 1-sided error with $q(\varepsilon/2) + O(1/\varepsilon)$ queries.

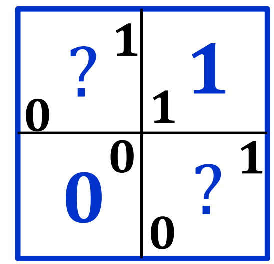
Partial Learner of Monotone functions $f: [n]^2 \to \{0, 1\}$

Lemma

There is an ε-partial learner for the class of monotone Boolean functions over $[n]^2$ that makes $O(1/\varepsilon)$ queries.

Idea:

- Divide the grid into quarters.



- Query the bottom left and the top right corner for each quarter.
- If the value of the function is NOT determined by the corners, recurse.

Details: Keep a quad tree and stop at $\log \frac{1}{\varepsilon} + 1$ levels.

- If $\geq 2^{j+1}$ nodes at level j are ?, fail.

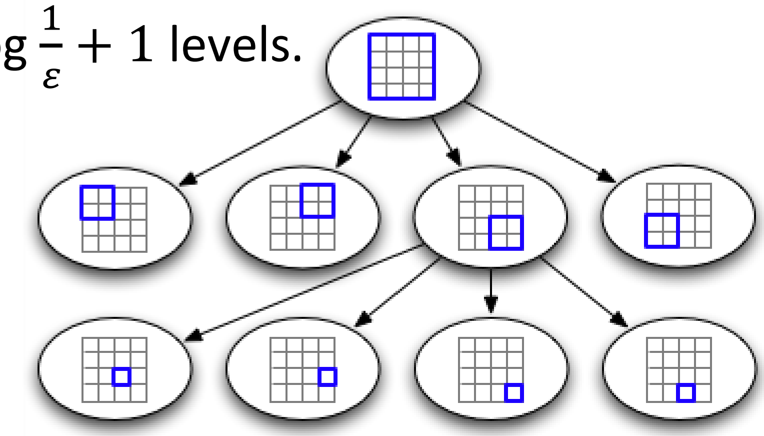


Image credit: Pradeep Pujari

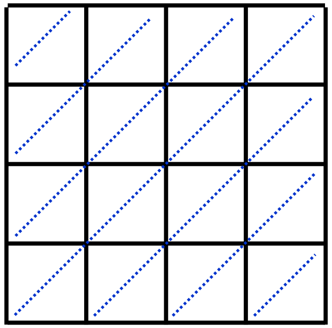
Correctness of the Learner

Claim

If the input function is monotone, level j will have fewer than 2^{j+1} nodes ?.

Proof:

$\boldsymbol{m = 2^j}$



$\boldsymbol{m = 2^j}$

Monotonicity Testers: Running Time

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Monotonicity Testers: Running Time

f	L_0	L_p
$[n] \to [0,1]$	$\Theta\left(\frac{\log n}{\boldsymbol{\varepsilon}}\right)$ [Ergün Kannan Kumar Rubinfeld Viswanathan 00, Fischer 04]	$\Theta\left(\frac{1}{\boldsymbol{\varepsilon}^p}\right)$
$[n]^d \to [0,1]$	$\Theta\left(\frac{d \cdot \log n}{\boldsymbol{\varepsilon}}\right)$ [Chakrabarty Seshadhri 13]	$O\left(\frac{d}{\boldsymbol{\varepsilon}^p} \log \frac{d}{\boldsymbol{\varepsilon}^p}\right)$ $\Omega\left(\frac{1}{\boldsymbol{\varepsilon}^p} \log \frac{1}{\boldsymbol{\varepsilon}^p}\right)$ for $d = 2$ nonadaptive 1-sided error

* Hiding some $\log 1/\varepsilon$ dependence

Distance Approximation and Tolerant Testing

Approximating L_1-distance to monotonicity $\pm\varepsilon$ $w.p. \geq 2/3$

f	L_0	L_1
$[n] \to [0,1]$	polylog $n \cdot \left(\frac{1}{\varepsilon}\right)^{O(1/\varepsilon)}$ [Saks Seshadhri 10]	$\Theta\left(\frac{1}{\varepsilon^2}\right)$

- Time complexity of tolerant L_1-testing for monotonicity is

$$O\left(\frac{\varepsilon_2}{(\varepsilon_2 - \varepsilon_1)^2}\right).$$

Open Problems

- Our L_1-tester for monotonicity is nonadaptive, but adaptivity helps for Boolean range.

Is there a better adaptive tester?

- All our algorithms for L_p-testing for $p \geq 1$ were obtained directly from L_1-testers.

Can one design better algorithms by working directly with L_p-distances?

- We designed tolerant tester only for monotonicity (d=1,2).

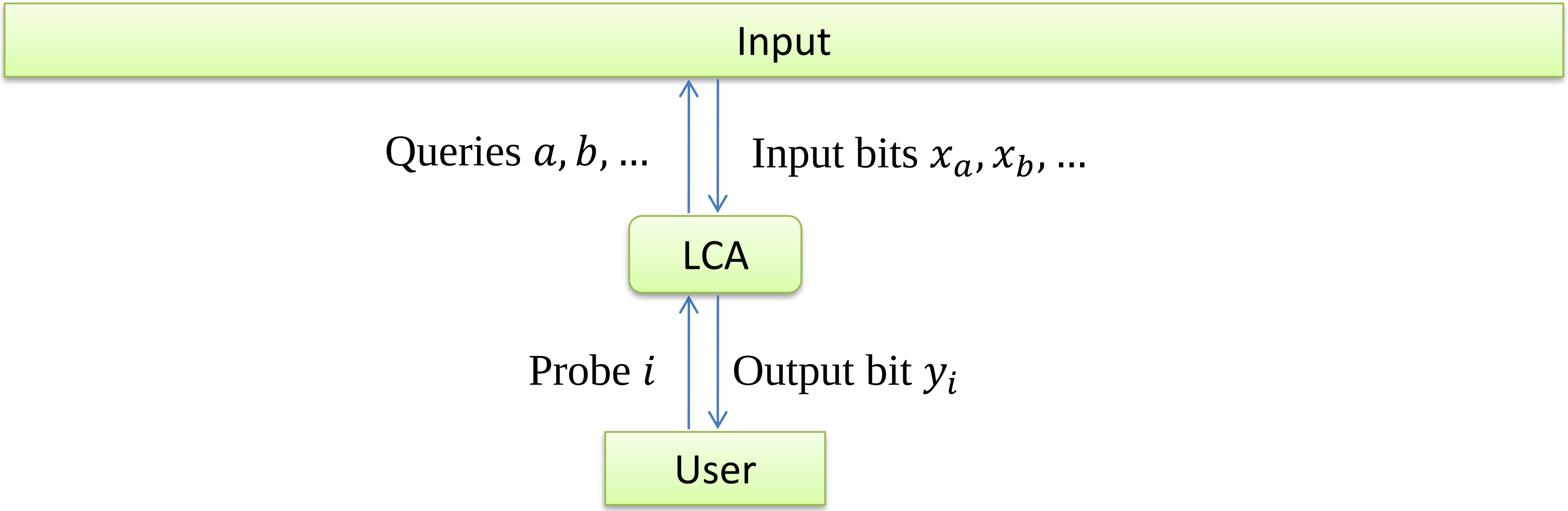
Tolerant testers for higher dimensions?

Other properties?

Local Computation Algorithms (LCAs)

Motivation: to have sublinear-time algorithms for problems with long output

- User should be able to ``probe'' bits of the output.



- If there are multiple possible outputs, LCA should be giving answers consistent with one.
- The order of the probes should not affect the answers (instantiations of LCA should be able to consistently answer probes in parallel)
- They can have access to the same random string.
- [Rubinfeld, Tamir, Vardi, Xie 11]

Maximal Independent Set (MIS)

For a graph $G = (V, E)$, a set $M \subseteq V$ is a maximal independent set if

- M is independent: $\forall u, v \in M$, the pair $(u, v) \notin E$
- M is maximal: no larger independent set contains M as a subset.

Example:

- MIS can be found in poly time by greedily adding vertices to M and removing them and their neighbors from consideration.
- It is NP-hard to compute a maximum independent set.

Goal: An LCA for MIS

- Given probe access to a graph G of maximum degree Δ, provide query access to an MIS M:

  in-MIS(v): Is v in M?

Main idea: modify an existing distributed algorithm for MIS.

Based on Ronitt Rubinfeld's and Sepehr Assadi's lecture notes

Distributed LOCAL Model

- The input graph is a communication network; each node is a processor.
- In each round:
  - Communication: each vertex can send any message to each neighbor (possibly different messages to different neighbors).
  - Computation: each vertex can decide on its actions for the next round, based on received messages.
- At the end of the last round, each vertex decides on its final status (e.g., whether it is in the MIS M)
- Goal: to minimize the number of rounds.

(A variant of) Luby's MIS Algorithm for the LOCAL Model

1. Initialize $Active(v) = True; M(v) = False$ for all $v \in V$.
2. For each (out of R) rounds, all vertices v run the following in parallel:
   a. Vertex v selects itself with probability $\frac{1}{2\Delta}$
   b. If $Active(v) = True$, v is selected, and no neighbor of v is selected then set $M(v) = True$ and $Active(u) = False$ $\forall u \in \{v\} \cup N(v)$

Correctness of Luby's Algorithm

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Correctness Theorem

Let M be the set of vertices for which $M(v) = True$.

1. After every round, M is an independent set
2. When $Active(v) = False$ for all $v \in V$ then M is an MIS.

Proof:

Analyzing the Number of Rounds

Termination Theorem

Fix $v \in V$ and round $R \geq 1$. Then

$\Pr[Active(v) = True$ after R rounds of Luby's algorithm$] \leq \exp\left(-\frac{R}{4\Delta}\right)$

Proof: For each $v \in V$ and round $r \geq 1$, define the following events.

$A_r(v)$: the event that $Active(v) = True$ after round r

$S_r(v)$: the event that v is selected in round r

$M_r(v)$: the event that v is added to M in round r

$$\Pr\left[\overline{A_r(v)} \;\middle|\; A_{r-1}(v)\right] \geq \Pr[\, M_r(v) \mid A_{r-1}(v)\,]$$

If v is added to M, it is no longer active

$$= \Pr\left[S_r(v) \wedge \forall u \in N(v): \overline{S_r(v)}\right]$$

$$= \Pr[S_r(v)] \cdot \Pr\left[\forall u \in N(v): \overline{S_r(v)}\right]$$

$$\geq \Pr[S_r(v)] \cdot \left(1 - \sum_{u \in N(v)} \Pr[S_r(u)]\right)$$

By union bound

$$\geq \frac{1}{2\Delta} \cdot \left(1 - \Delta \cdot \frac{1}{2\Delta}\right) = \frac{1}{4\Delta}$$

If v is active, it will be deactivated in this round w.p. $\geq \frac{1}{4\Delta}$

Analyzing the Number of Rounds

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Proof: For each $v \in V$ and round $r \geq 1$, define the following events.

$A_r(v)$: the event that $Active(v) = True$ after round r

$$\Pr\left[\overline{A_r(v)} \mid A_{r-1}(v)\right] \geq \frac{1}{4\Delta}$$

$$\Pr[A_R(v)] = \prod_{r=1}^{R} \Pr[A_r(v) \mid A_{r-1}(v)]$$

By Product Rule

$$\leq \left(1 - \frac{1}{4\Delta}\right)^R \leq \exp\left(-\frac{R}{4\Delta}\right)$$

Conclusion: Set $R = 8\Delta \cdot \ln n$.

- Then a specific vertex remains active after R rounds w.p. at most $1/n^2$
- By a union bound, no vertex remains active w.p. at least $1 - 1/n$