Sublinear Algorithms

LECTURE 10

Last time

- Multipurpose sketches
- Count-min and count-sketch
- Range queries, heavy hitters, quantiles

Today

- Limitations of streaming algorithms
- Communication complexity

Recall: Frequency Moments Estimation

Input: a stream $\langle a_1, a_2, \dots, a_m \rangle \in [n]^m$

- The frequency vector of the stream is $f = (f_1, \dots, f_n)$, where f_i is the number of times i appears in the stream
- The p-th frequency moment is $F_p = ||f||_p^p = \sum_{i=1}^n f_i^p$

  F_0 is the number of nonzero entries of f (# of distinct elements)

  $F_1 = m$ (# of elements in the stream)

  $F_2 = ||f||_2^2$ is a measure of non-uniformity

  used e.g. for anomaly detection in network analysis

  $F_\infty = \max_i f_i$ is the most frequent element

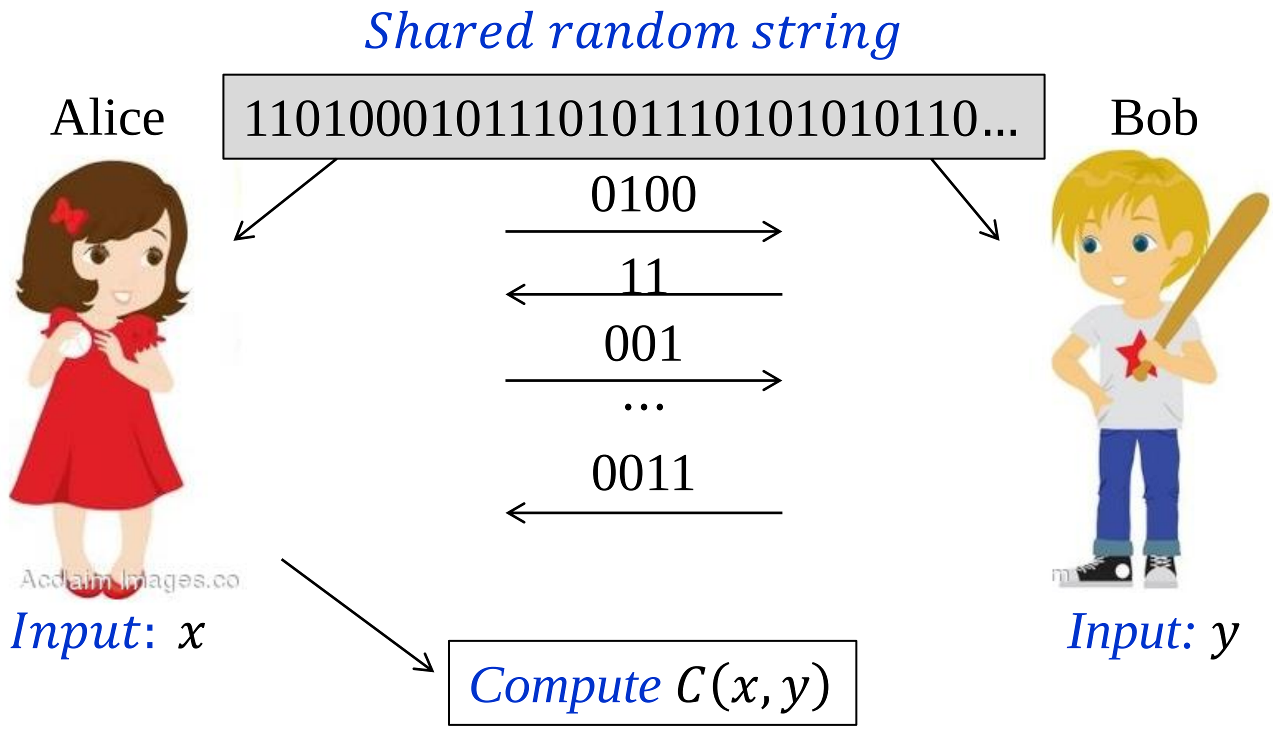
We obtained streaming algorithms for F_0, F_1, F_2.

What about F_3 to F_∞?

Communication Complexity

A Method for Proving Lower Bounds

(Randomized) Communication Complexity



Goal: minimize the number of bits exchanged.

- Communication complexity of a protocol is the maximum number of bits exchanged by the protocol.
- Communication complexity of a function C, denoted $R(C)$, is the communication complexity of the best protocol for computing C.

Partially based on slides by Eric Blais

Example: Set Disjointness $DISJ_k$

Alice

11010001011101011101010110...

Bob

Input: $S \subseteq [n]$, $|S| = k$.

Input: $T \subseteq [n]$, $|T| = k$

Compute $DISJ_k(S,T)$

$$= \begin{cases} \boldsymbol{accept} & \text{if } S \cap T = \emptyset \\ \boldsymbol{reject} & \text{otherwise} \end{cases}$$

Theorem [Kalyanasundaram Schmitger 92, Razborov 92]

$$R(\mathrm{DISJ}_k) \geq \Omega(k) \text{ for all } k \leq \frac{n}{2}.$$

One-Way Communication Complexity

Shared random string

Alice 1101000101110101110101010110... Bob

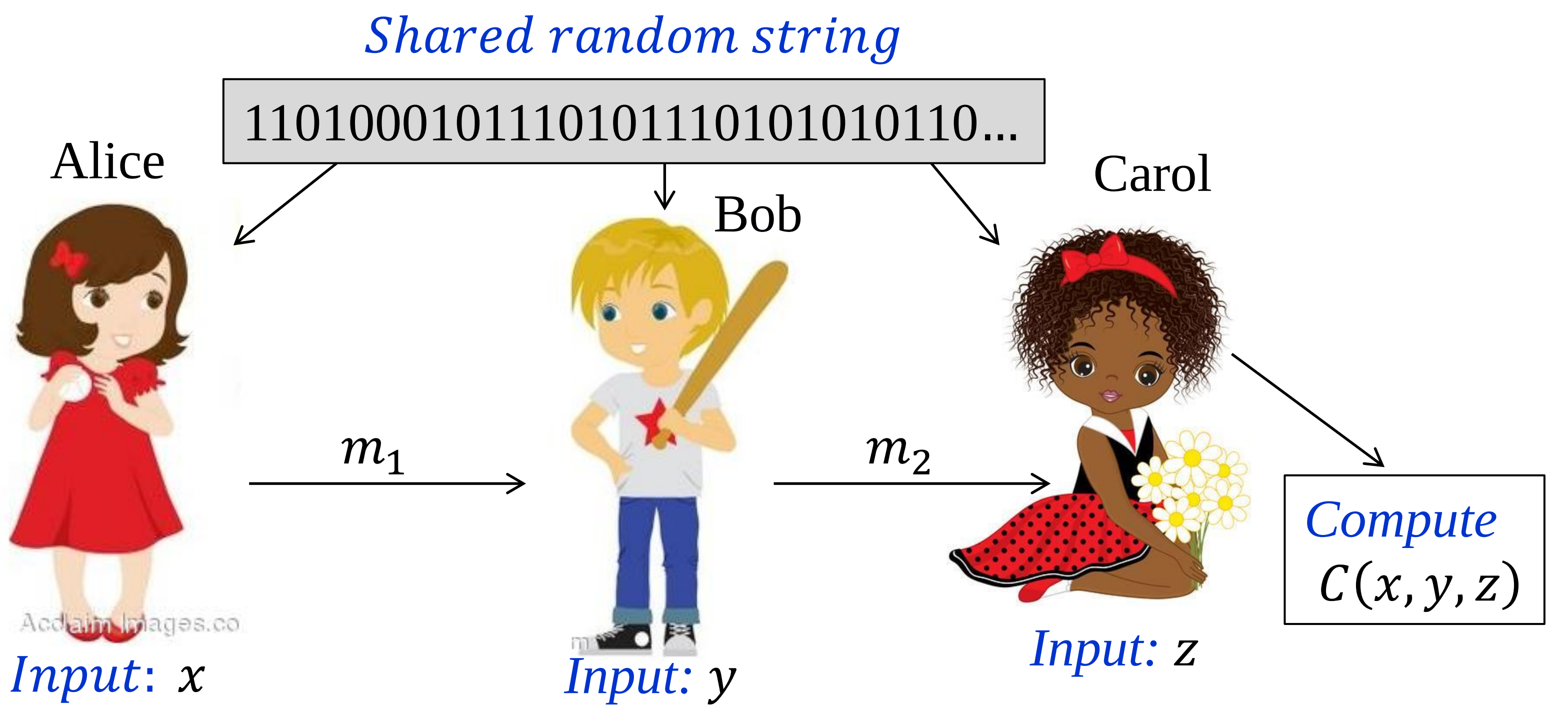
Input: x $\xrightarrow{m_1}$ *Input:* y

Compute $C(x, y)$

Goal: minimize the number of bits Alice sends to Bob.

One-way communication complexity of a function C, denoted $R^{\rightarrow}(C)$, is the communication complexity of the best one-way protocol for computing C.

3-Player One-Way Communication Complexity



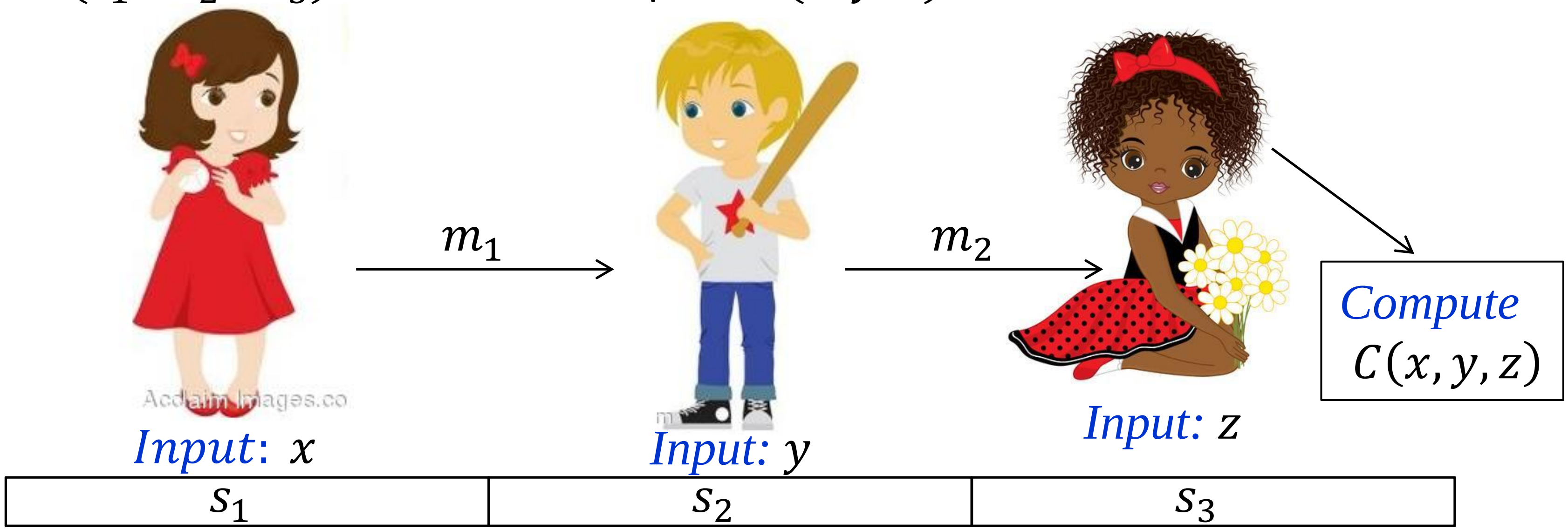
Goal: minimize $|m_1| + |m_2|$.

- Require correct output w.p. at least 2/3 over the random string

Converting Streaming Algorithm to CC Protocol

Let $\mathcal{P}$ be a streaming problem.

- Suppose there is a transformation $x \to s_1, y \to s_2, z \to s_3$ such that $\mathcal{P}(s_1 \circ s_2 \circ s_3)$ suffices to compute $C(x, y, z)$



An s-bit algorithm A for $\mathcal{P}$ gives a $2s$-bit protocol for C

- Alice runs A on s_1 and sends memory state, m_1, to Bob
- Bob instantiates A with m_1, runs A on s_2, sends memory state, m_2, to Carol
- Carol instantiates A with m_2, runs A on s_3 to get $\mathcal{P}(s_1 \circ s_2 \circ s_3)$ and computes $C(x, y, z)$

Based on Andrew McGregor's slides: https://people.cs.umass.edu/~mcgregor/711S18/lowerbounds-1.pdf

Converting Streaming Algorithm to CC Protocol

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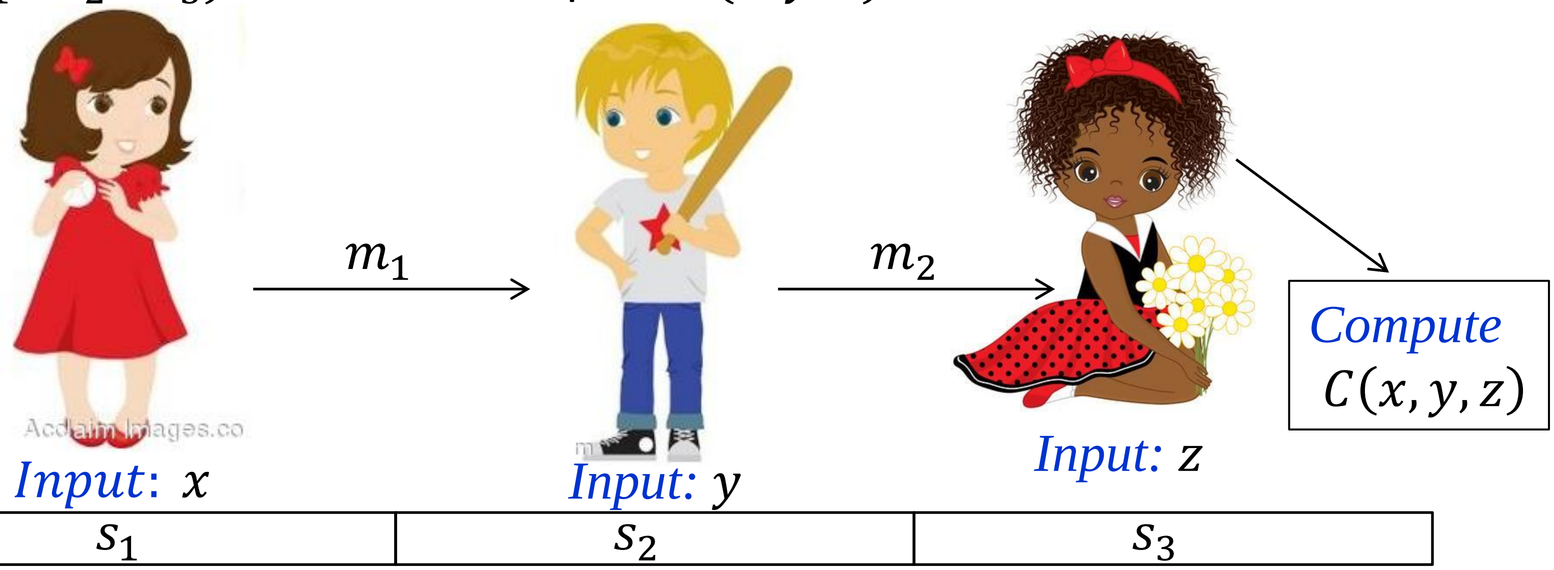

An s-bit algorithm A for $\mathcal{P}$ gives a $2s$-bit protocol for C

- If there are p players than the protocol uses $(p-1)s$ bits
- A lower bound L for computing C implies $b = \Omega\left(\frac{L}{p}\right)$

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A lower bound using CC method

Approximating F_∞

Application: Approximating F_∞

Theorem

Every algorithm that computes $4/3$-approximation of F_∞ (w.p. ≥2/3) needs $\Omega(n)$ space.

Proof: Reduction from Set Disjointness

On input $x, y \in \{0,1\}^n$, players generate $s_1 = \{j: x_j = 1\}$ and $s_2 = \{j: y_j = 1\}$

Example: $(0\ 0\ 1\ 1\ 0\ 0)$ $(1\ 0\ 1\ 0\ 1\ 0)$ $\rightarrow \langle 3,4; 1,3,5 \rangle$

- Then $F_\infty = 1$ if x, y represent disjoint sets, and $F_\infty = 2$, otherwise.

  Output $\leq 4/3$ Output $\geq 3/2$

- An s-space algorithm implies an s-bit protocol:

$$s = \Omega(n)$$

↑ by communication complexity of *Set Disjointness*

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A lower bound using CC method

Computing the median of a stream

Index

- Alice gets an n-bit string x, and Bob gets an index $j \in [n]$.
- Define $Index(x, j) = x_j$.
- One-way communication complexity of $Index(x, j)$ is $\Omega(n)$

Application: Finding the Median of a Stream

Theorem

Every algorithm that computes the median of an $(2n-1)$-element stream exactly (w.p. ≥2/3) needs $\Omega(n)$ space.

Proof: Reduction from Index.

- On input $x \in \{0,1\}^n$, Alice generates $s_1 = \{2i + x_i : i \in [n]\}$

  Example: 0 0 1 1 0 1 1 $\rightarrow \langle 2,4,7,9,10,13,15 \rangle$

- On input $j \in [n]$, Bob generates

  $$s_2 = \{n - j \text{ copies of } 0 \text{ and } j - 1 \text{ copies of } 2n + 2\}$$

  Example: $j = 2$ $\rightarrow \langle 0,0,0,0,0,16 \rangle$

- Then $median(s_1 \circ s_2) = 2j + x_j$ and Index$(x,j) = 2j + x_j \; mod \; 2$
- An s-space algorithm implies an s-bit protocol:

  $$s = \Omega(n)$$

by 1-way communication complexity of *Index*

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A lower bound using CC method

Approximating Frequency Moments

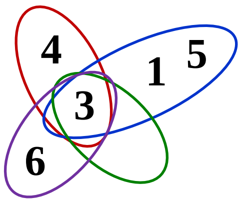
[Bar-Yossef, Jayram, Kumar, Sivakumar 04]

Multi-party Set Disjointness

- Consider a $p \times n$ binary matrix M where each column has weight 0, 1 or p

Example:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix}$$



- The input of player i is row i of M

$$DISJ^{(p)}(M) = \begin{cases} 0 & \text{if there is a column of 1s} \\ 1 & \text{otherwise} \end{cases}$$

- Communication complexity of $DISJ^{(p)}(M)$ is $\Omega\left(\frac{n}{p}\right)$

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Application: Frequency Moments for k > 2

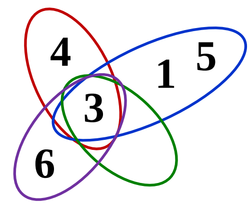
Thm. Every algorithm that 2-approximaes F_k (w.p. $\geq 2/3$) needs $\Omega\left(n^{1-\frac{2}{k}}\right)$ space

Proof: Reduction from multi-party Set Disjointness

- On input $M \in \{0,1\}^{p\times n}$, player i generates $s_i = \{j: M_{ij} = 1\}$

Example:

$$\begin{pmatrix} 0 & 0 & 1 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 1 \end{pmatrix} \rightarrow \langle 3,4;\ 1,3,5;\ 3;\ 3,6 \rangle$$



- If all columns have weight 0 or 1 then $F_k = \sum_{i=1}^{n} f_i^k \leq n$
- If there is a column of weight p then $F_k \geq p^k$
- A 2-approximation of F_k distinguishes the cases if $p^k > 4n \Leftrightarrow p > (4n)^{\frac{1}{k}}$
- An s-space algorithm implies $s(p-1)$-bit protocol:

$$s = \Omega\left(\frac{n}{p^2}\right) = \Omega\left(\frac{n}{(4n)^{\frac{2}{k}}}\right) = \Omega\left(n^{1-\frac{2}{k}}\right)$$

by communication complexity of $DISJ^{(p)}$

for constant k

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A lower bound using CC method

Distinct Elements

Gap Hamming

- Alice and Bob get n-bit strings x and y, respectively.

- Hamming distance $Ham(x, y)$ is the number of positions on which x and y differ.

- Output: $Ham(x, y)$ with additive error $\sqrt{n}$ w.p. $\geq 2/3$

- Communication complexity of $Ham(x, y)$ is $\Omega(n)$

  even when $|x|$ and $|y|$ are known to both players

Based on Andrew McGregor's slides: https://people.cs.umass.edu/~mcgregor/711S18/lowerbounds-1.pdf

Application: Distinct Elements

Thm. Every algorithm $(1+\varepsilon)$-approximating F_0 (w.p. $\geq 2/3$) needs $\Omega(1/\varepsilon^2)$ space

Proof: Reduction from Gap Hamming

On input $x, y \in \{0,1\}^n$, players generate $s_1 = \{j : x_j = 1\}$ and $s_2 = \{j : y_j = 1\}$

Example: $(0\ 0\ 1\ 1\ 0\ 0)$ $(1\ 0\ 1\ 0\ 1\ 0)$ $\rightarrow \langle 3,4; 1,3,5 \rangle$

- Then $2F_0 = |x| + |y| + Ham(x, y)$
- When $|x|$ is known to Bob, $(1+\varepsilon)$-approximation of F_0 gives an additive approximation to $\text{Ham}(x, y)$

$$\varepsilon \cdot \frac{|x| + |y| + Ham(x,y)}{2} \leq \varepsilon n \leq \sqrt{n}$$

for $\varepsilon \leq 1/\sqrt{n}$

- An s-space algorithm implies an s-bit protocol:

$$s = \Omega(n) = \Omega\left(\frac{1}{\varepsilon^2}\right)$$

by communication complexity of *Gap Hamming*

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Proving New CC Lower Bounds

Lower Bound for Gap Hamming

A Lower Bound for Gap Hamming

- Alice and Bob get n-bit strings x and y, respectively.
- Hamming distance $Ham(x, y)$ is the number of bits on which x and y differ.
- Output: $Ham(x, y)$ with additive error $\sqrt{n}$ w.p. $\geq 2/3$

Theorem

One way communication complexity $R^{\rightarrow}$ of $Ham(x, y)$ is $\Omega(n)$

Proof: Reduction from Index

- Alice gets $z \in \{0,1\}^t$, Bob gets $j \in [t]$
- Assumption: $|z| = t/2$ and $|z|$ is odd
- Alice and Bob pick uniformly random $r \in \{-1,1\}^t$ using public randomness
- Let $r \cdot z$ denote $\sum_{i \in [n]} r_i z_i$
- Alice computes $sgn(r \cdot z)$; Bob computes $sgn(r_j)$

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Reduction from Index to Gap Hamming

Proof: Alice gets $z \in \{0,1\}^t$, Bob gets $j \in [t]$

- Assumption: $|z| = t/2$ and $|z|$ is odd
- Alice and Bob pick uniformly random $r \in \{-1,1\}^t$ using public randomness
- Alice computes $sgn(r \cdot z)$; Bob computes $sgn(r_j)$

Lemma

There exists a constant $c > 0$ such that

$$\Pr\left[sgn(r \cdot z) = sgn(r_j)\right] = \begin{cases} \mathbf{1/2} & \text{if } z_j = 0 \\ \mathbf{1/2 + c/\sqrt{t}} & \text{otherwise} \end{cases}$$

- Alice and Bob repeat $n = 25t/c^2$ times to construct each bit
- If $z_j = 0$ then $\mathbb{E}[Ham(x,y)] =$
- If $z_j = 1$ then $\mathbb{E}[Ham(x,y)] =$

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Proof of Lemma

Lemma

There exists a constant $c > 0$ such that

$$\Pr[sgn(r \cdot z) = sgn(r_j)] = \begin{cases} \mathbf{1/2} & \text{if } z_j = 0 \\ \mathbf{1/2} + \boldsymbol{c}/\sqrt{\boldsymbol{t}} & \text{otherwise} \end{cases}$$

Proof: Let A denote the event in the lemma

- If $z_j = 0$
- If $z_j = 1$

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