LECTURE 24

Last time

- PAC learning and VC-dimension
- The sample complexity of PAC learning

Today

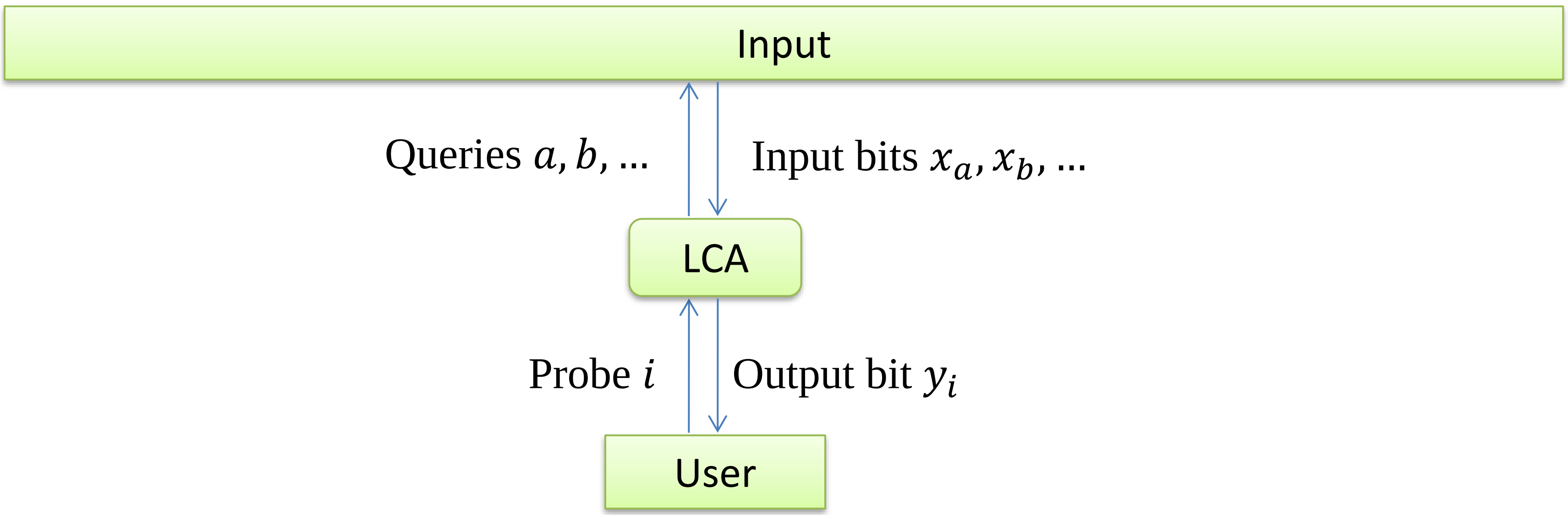
- Local Computation Algorithms (LCAs)
- Distributed LOCAL model
- Maximal Independent Set (MIS)

Project Reports are due Thursday, April 24

Local Computation Algorithms (LCAs)

Motivation: to have sublinear-time algorithms for problems with long output

- User should be able to ``probe'' bits of the output.



- If there are multiple possible outputs, LCA should be giving answers consistent with one.
- The order of the probes should not affect the answers (instantiations of LCA should be able to consistently answer probes in parallel)
- They can have access to the same random string.
- [Rubinfeld, Tamir, Vardi, Xie 11]

Maximal Independent Set (MIS)

For a graph $G = (V, E)$, a set $M \subseteq V$ is a maximal independent set if

- M is independent: $\forall u, v \in M$, the pair $(u, v) \notin E$
- M is maximal: no larger independent set contains M as a subset.

Example:

- MIS can be found in poly time by greedily adding vertices to M and removing them and their neighbors from consideration.
- It is NP-hard to compute a maximum independent set.

Goal: An LCA for MIS

- Given (adjacency lists) query access to a graph G of maximum degree Δ, provide probe access to an MIS M:

  in-MIS(v): Is v in M?

Today: an LCA by [Rubinfeld, Tamir, Vardi, Xie 11] with run time $\Delta^{O(\Delta \log \Delta)} \cdot \log n$

Main idea: modify an existing distributed algorithm for MIS.

Based on Ronitt Rubinfeld's and Sepehr Assadi's lecture notes

Distributed LOCAL Model

- The input graph is a communication network; each node is a processor.
- In each round:
  - Communication: each vertex can send any message to each neighbor (possibly different messages to different neighbors).
  - Computation: each vertex can decide on its actions for the next round, based on received messages.
- At the end of the last round, each vertex decides on its final status (e.g., whether it is in the MIS M)
- Goal: to minimize the number of rounds.

(A variant of) Luby's MIS Algorithm for the LOCAL Model

1. Initialize $Active(v) = True$; $M(v) = False$ for all $v \in V$.
2. For each (out of R) rounds, all vertices v run the following in parallel:
   a. Vertex v selects itself with probability $\frac{1}{2\Delta}$
   b. Vertex v wins if v is selected, and no neighbor of v is selected
   c. If v won and $Active(v) = True$, then set $M(v) = True$ and $Active(u) = False$ $\forall u \in \{v\} \cup N(v)$

Correctness of Luby's Algorithm

(A variant of) Luby's MIS Algorithm for the LOCAL Model

1. Initialize $Active(v) = True$; $M(v) = False$ for all $v \in V$.
2. For each (out of R) rounds, all vertices v run the following in parallel:
   a. Vertex v selects itself with probability $\frac{1}{2\Delta}$
   b. Vertex v wins if v is selected, and no neighbor of v is selected
   c. If v won and $Active(v) = True$, then set $M(v) = True$ and $Active(u) = False\ \forall u \in \{v\} \cup N(v)$

Correctness Theorem

Let M be the set of vertices for which $M(v) = True$.

1. After every round, M is an independent set
2. When $Active(v) = False$ for all $v \in V$ then M is an MIS.

Proof:

Analyzing the Number of Rounds

Termination Theorem

Fix $v \in V$ and round $R \geq 1$. Let $L(v)$ be the event that v lost in all R rounds.
Then $\Pr[Active(v) = True$ after R rounds of Luby's algorithm$]$

$$\leq \Pr[L(v)] \leq \exp\left(-\frac{R}{4\Delta}\right).$$

Proof: For each $v \in V$ and round $r \geq 1$, define the following events.

$S_r(v)$: the event that v is selected in round r

$W_r(v)$: the event that v *wins* round r, i.e., v is the only selected vertex in $\{v\} \cup N(v)$

$$\Pr[W_r(v)] = \Pr\left[S_r(v) \wedge \forall u \in N(v): \overline{S_r(u)}\right]$$

$$= \Pr[S_r(v)] \cdot \Pr\left[\forall u \in N(v): \overline{S_r(u)}\right]$$

Events $S_r(v)$ are independent

$$\geq \Pr[S_r(v)] \cdot \left(1 - \sum_{u \in N(v)} \Pr[S_r(u)]\right)$$

By a union bound

$$\geq \frac{1}{2\Delta} \cdot \left(1 - \Delta \cdot \frac{1}{2\Delta}\right) = \frac{1}{4\Delta}$$

Analyzing the Number of Rounds

Termination Theorem

Fix $v \in V$ and round $R \geq 1$. Let $L(v)$ be the event that v lost in all R rounds.
Then $\Pr[Active(v) = True$ after R rounds of Luby's algorithm$]$

$$\leq \Pr[L(v)] \leq \exp\left(-\frac{R}{4\Delta}\right).$$

Proof: $W_r(v)$: the event that v *wins* round r

- $\Pr[W_r(v)] \geq \frac{1}{4\Delta}$
- Events $W_r(v)$ are independent for different rounds
- The probability that v is active after R rounds is at most

$$\Pr[L(v)] \leq \prod_{r=1}^{R} \Pr\left[\overline{W_r(v)}\right] \leq \left(1 - \frac{1}{4\Delta}\right)^R \leq \exp\left(-\frac{R}{4\Delta}\right)$$

- If v wins, it is no longer active

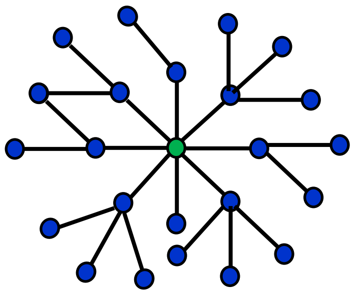
$\square$

Conclusion: Set $R = 8\Delta \cdot \ln n$.

- Then a specific vertex remains active after R rounds w.p. at most $1/n^2$
- By a union bound, no vertex remains active w.p. at least $1-1/n$

Converting Luby's MIS Algorithm to LCA

- Key observation: What happens to vertex v in R rounds depends only on R-hop neighborhood of v



2-hop neighborhood

- If we simulate Luby's algorithm for $R = \Theta(\Delta \log n)$ rounds, we need to consider R-hop neighborhood of v, which takes $\Delta^{\Theta(\Delta \log n)} = \Omega(n)$ time.
- Idea 1: Simulate it for $R = \Theta(\Delta \log \Delta)$ rounds instead (no dependence on n)
- Idea 2: Prove that, at the end, active vertices form small connected components. (We say that the graph is shattered.)
- For each probe v, if its MIS status has not been decided (i.e., v is still active) after R rounds, we will find MIS for its connected component deterministically.

LCA for MIS

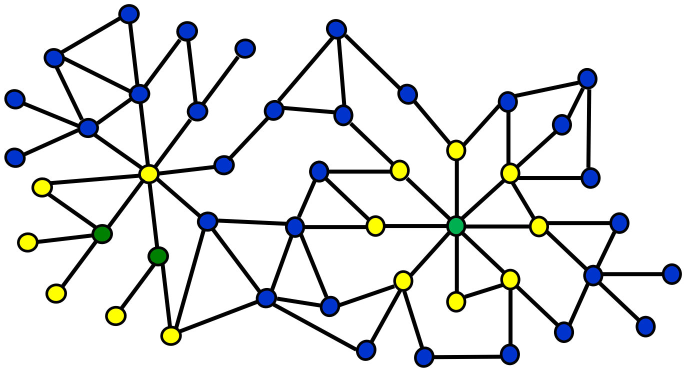
LubyStatus(v, R)

1. Simulate Luby's algorithm on vertex v for R rounds
2. If $Active(v) = False$ then
3. if $M(v) = True$, return IN-MIS; otherwise, return NOT-IN-MIS
4. else return ACTIVE

Answer Probe in-MIS(v)

1. Set $R = 12\Delta \cdot \ln(2\Delta)$
2. Compute $status \leftarrow$ LubyStatus(v, R)
3. If $status$ is IN-MIS or NOT-IN-MIS, return $status$
4. Otherwise, find the connected component C_v of v as follows:
5. Run DFS on v
6. For every visited node u, compute LubyStatus(u, R)
7. Continue DFS only on active nodes
8. Compute lexicographically first MIS of C_v greedily, ordering vertices according to their ID.
9. Return whether v belongs to MIS of C_v

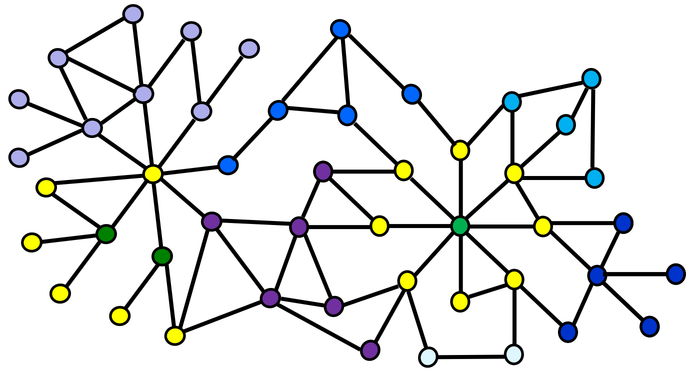
Correctness



The output is an independent set

- Luby’s algorithm maintains an independent set.
- Active vertices are not adjacent to vertices already in MIS.

Correctness



The output is an independent set

- Luby's algorithm maintains an independent set.
- Active vertices are not adjacent to vertices already in MIS.
- If $C_u \neq C_v$ then $(u, v) \notin E$, so when we add independent sets for connected components, the resulting set is independent

The output is a maximal independent set

- Each deactivated vertex that is not in the output M is adjacent to a vertex in M, so it cannot be added.
- If v was in a connected component C_v, but is not in M, it cannot be added because M includes an MIS for C_v.

Running Time

Runtime Theorem

Important: run time applies to all probes simultaneously

W.p. ≥2/3 over random strings, each probe in-MIS(v) is answered in $\Delta^{O(\Delta\cdot\log\Delta)}\cdot\log n$ time when the algorithm uses the chosen random string.

Lemma

For each v, it take time $\Delta^{O(\Delta\cdot\log\Delta)}\cdot|C_v|$ to answer probe in-MIS(v).

Proof: Consider running LubyStatus(u, R) for some $u \in V$.

- There are at most Δ^R vertices in the R-hop neighborhood of u.
- Since $R = O(\Delta\log\Delta)$, the running time is $\Delta^{O(\Delta\cdot\log\Delta)}$.

To answer probe in-MIS(v), we might run LubyStatus(u, R) on nodes in C_v and their neighbors, resulting in time at most

$$\Delta^{O(\Delta\cdot\log\Delta)}\cdot \mathrm{O}(\Delta)\cdot|C_v| = \Delta^{O(\Delta\cdot\log\Delta)}\cdot|C_v|.$$

It remains to analyze $|C_v|$.

Analyzing the Sizes of Connected Components

For each $v \in V$, define $A(v)$: the event that $Active(v) = True$ after round R

- By Termination Theorem, for each $v \in V$,

$$\Pr[A(v)] \leq \exp\left(-\frac{R}{4\Delta}\right) = \exp\left(-\frac{12\Delta \cdot \ln(2\Delta)}{4\Delta}\right) = \frac{1}{8\Delta^3}$$

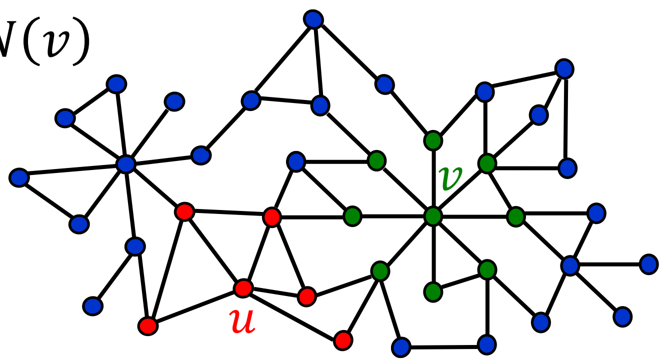
- One difficulty is that events $A(v)$ are not independent.

For each $v \in V$, define $L(v)$: the event that v is a loser (in all R rounds)

$\Pr[L(v)] \leq \frac{1}{8\Delta^3}$, as before.

Claim. Events $L(v)$ are independent for all vertices u, v at distance at least 3.

- $L(v)$ is only a function of randomness at $\{v\} \cup N(v)$
- Sets $\{u\} \cup N(u)$ and $\{v\} \cup N(v)$ are disjoint



Idea: Let H be the subgraph of G induced by losers.

We will show: if H has a large CC then it also has many ``independent'' nodes

Graph $G^{(3)}$

- Let $d_G(u, v)$ denote the distance from u to v in G
- Let $G^{(3)}$ be a graph on nodes $V(G)$ with $(u, v) \in E\left(G^{(3)}\right)$ iff $d_G(u, v) = 3$
- Max degree in $G^{(3)}$ is at most Δ^3
- For $S \subseteq V$, let $G[S]$ denote the induced subgraph of G on S

Big-Tree Claim

If $H[S]$ is connected then $H^{(3)}[S]$ contains a tree with a vertex set T as a subgraph, where $|\mathrm{T}| \geq \frac{|S|}{\Delta^2+1}$ and $d_H(u, v) \geq 3$ for all nodes $u, v \in T$.

Proof: We construct T greedily:

1. Pick an arbitrary $v \in S$
2. Repeat until no node remains in S:
3. Move v from S to T; remove all u with $d_H(u, v) < 3$ from S
4. Pick a new node $v \in S$ such that $d_H(u, v) = 3$ for some $u \in T$

For each node added to T, we exclude $\leq \Delta^2$ nodes from its 2-hop neighborhood, so T has the desired size.

Counting Trees in $G^{(3)}$

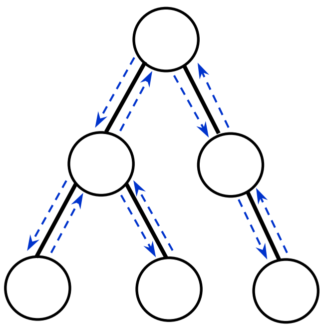
Tree-Counting Claim

For $s \geq 1$, let $\mathcal{T}_s$ denote the set of all s-node trees that are subgraphs of $G^{(3)}$.

Then $|\mathcal{T}_s| \leq n \cdot (4\Delta^3)^s$.

Proof: We enumerate trees in $\mathcal{T}_s$ using the following steps.

1. Chose the root. n choices
2. Choose an unlabeled s-node rooted tree by choosing its DFS sequence represented as $2(s-1)$-bit string. $\leq 2^{2(s-1)} < 4^s$ choices
3. Label the tree starting from the root in the order given by the DFS sequence. To go from a parent to a child, pick one of $\leq \Delta^3$ neighbors of the parent in $G^{(3)}$ as its child. $\leq \Delta^{3(s-1)} < \Delta^{3s}$ choices



The Size of Connected Components

$$L(T) = \wedge_{v \in T} L(v)$$
the event that all vertices in T are losers

- Let $s = \log \frac{n}{3}$
- Let $\mathcal{T}_s^* = \left\{T \subseteq V : |T| = s, G^{(3)}[T] \text{ contains a tree}, d_H(u,v) \geq 3\ \forall u, v \in T\right\}$
- The probability that there is a set $\mathrm{T} \in \mathcal{T}_s^*$ where all nodes are losers is

$$\leq \sum_{T \in \mathcal{T}_s^*} \Pr[L(T)] \leq |\mathcal{T}_s^*| \cdot \left(\frac{1}{(8\Delta)^3}\right)^s \leq n \cdot (4\Delta^3)^s \cdot \left(\frac{1}{8\Delta^3}\right)^s = n \cdot \frac{1}{2^s} = \frac{1}{3}$$

- But if there are no such trees, all CCs in H have size

$$\leq (\Delta^2 + 1) \log \frac{n}{3} = O(\Delta^2 \log n)$$

- That is, with probability at least 2/3, each probe takes time

$$\Delta^{O(\Delta \log \Delta)} \cdot O(\Delta^2 \log n) = \Delta^{O(\Delta \log \Delta)} \cdot \log n$$

□

Currently best run time of LCA for MIS is $\boldsymbol{poly(\Delta) \cdot \log n}$ [Ghaffari 22]