CS585 Assignment 2 Written Part

Yida Xin

September 27, 2018

Exercise 1: Normalized Cross-Correlation

Part 1: Prove that $r(M, S) = r(M, aS + b)$.

We are provided that

$$r(M, S) := \frac{1}{n} \sum_{i=1}^{n} \frac{(s_i - \mu_S)(m_i - \mu_M)}{\sigma_S \sigma_M},$$

where $s_i \in S$ and $m_i \in M$ are respective brightness values of the ith pixel, μ_M and σ_M are the mean and the standard deviation of all pixels in the template M, and μ_S and σ_S are the mean and the standard deviation of all the pixels in the sub-image of the scene.

By the laws of probability, for a random variable X, we have

$$\mathbb{E}[aX + b] = a\mathbb{E}[X] + b,$$

and

$$
\begin{aligned}
\mathbb{V}[aX + b] &= \mathbb{E}[(aX + b)^2] - (\mathbb{E}[aX + b])^2 \\
&= \mathbb{E}[a^2 X^2 + 2abX + b^2] - (a\mathbb{E}[X] + b)^2 \\
&= a^2 \mathbb{E}[X^2] + 2ab\mathbb{E}[X] + b^2 - \left[a^2(\mathbb{E}[X])^2 + 2ab\mathbb{E}[X] + b^2\right] \\
&= a^2 \left(\mathbb{E}[X^2] - (\mathbb{E}[X])^2\right) \\
&= a^2 \mathbb{V}[X],
\end{aligned}
$$

meaning that the standard deviation of $aX + b$, σ_{aX+b}, is

$$\sigma_{aX+b} = \sqrt{\mathbb{V}[aX + b]} = \sqrt{a^2 \mathbb{V}[X]} = a \cdot \sigma_X.$$

Consequently, we have

$$
\begin{aligned}
r(M, aS + b) &= \frac{1}{n} \sum_{i=1}^{n} \frac{\left((a \cdot s_i + b) - (a \cdot \mu_S + b)\right)(m_i - \mu_M)}{\sigma_{aS+b} \sigma_M} \\
&= \frac{1}{n} \sum_{i=1}^{n} \frac{a \cdot (s_i - \mu_S)(m_i - \mu_M)}{a \cdot \sigma_S \sigma_M} \\
&= \frac{1}{n} \sum_{i=1}^{n} \frac{(s_i - \mu_S)(m_i - \mu_M)}{\sigma_S \sigma_M} \\
&= r(M, S).
\end{aligned}
$$

This concludes the proof.

Part 2: Explain why the linear invariance property of the normalized correlation coefficient, shown in part (a), could be useful for image analysis.

Because changing the brightness values of the pixels in any sub-image of the overall scene does not affect the NCC value, we can, in practical situations, increase or decrease the brightness of our images of interest at will.

Part 3: Explain why the property that the NCC value is between -1 and $+1$, inclusively, could be useful for image analysis.

As hinted in the name itself (Normalized Cross-*Correlation*), the fact that NCC values are between -1 and $+1$ makes it easy for us to tell whether the sub-image and our template are positively or negatively correlated, whether they are correlated at all, to what extent are they correlated, etc. — regardless of the brightness values of the pixels in the sub-images, as indicated in Part 2 above.

Part 4: Explain why the fact that $\mathbb{E}[r] = 0$ could be useful for image analysis.

Again, by the formula

$$\mathbb{V}[r] = \mathbb{E}[r^2] - (\mathbb{E}[r])^2,$$

the fact that $\mathbb{E}[r] = 0$ means that

$$\mathbb{V}[r] = \mathbb{E}[r^2].$$

This makes the computation of r's variance, and thus standard deviation, easy, which again enables us to easily tell how much our template correlates with a given sub-image.

Exercise 2: Circularity

Part 1. Compute *compactness* for each shape

(1) Compactness of "Square"

$$\text{compactness} = \frac{\text{perimeter}^2}{\text{area}} = \frac{(3+3+3+3)^2}{3 \times 3} = \frac{144}{9} = 16.$$

(2) Compactness of "Rectangle"

$$\text{compactness} = \frac{\text{perimeter}^2}{\text{area}} = \frac{(2+4+2+4)^2}{2 \times 4} = \frac{144}{8} = 18.$$

(3) Compactness of "Diamond"

$$\text{compactness} = \frac{\text{perimeter}^2}{\text{area}} = \frac{(5+5+5+5)^2}{1+3+5+3+1} = \frac{400}{13} \approx 30.77.$$

(4) Compactness of "Stick"

$$\text{compactness} = \frac{\text{perimeter}^2}{\text{area}} = \frac{(1+10+1+10)^2}{1 \times 10} = \frac{484}{10} = 48.4.$$

Part 2. Compute the second-moment measure for each shape

From class notes, we know that

$$E_{\min} := \frac{(a+c) - \sqrt{(a-c)^2 + b^2}}{2} \quad \text{and} \quad E_{\max} := \frac{(a+c) + \sqrt{(a-c)^2 + b^2}}{2},$$

where

$$a := \sum_x \sum_y (x - \bar{x})^2 \cdot B_{xy}$$

$$c := \sum_x \sum_y (y - \bar{y})^2 \cdot B_{xy}$$

$$b := 2 \sum_x \sum_y (x - \bar{x})(y - \bar{y}) \cdot B_{xy}.$$

For simplicity, I assume that, for each pixel of each given shape, $B_{xy} = 1$. I also assume that we start counting at 1, not 0.

(1) Second-moment Measure for "Square"

For "square," we can compute that

$$a = \sum_{x=1}^{3} \sum_{y=1}^{3} (x-2)^2 = 3 \sum_{x=1}^{3} (x-2)^2 = 6$$

$$c = \sum_{x=1}^{3} \sum_{y=1}^{3} (y-2)^2 = 3 \sum_{y=1}^{3} (y-2)^2 = 6$$

$$b = 2 \sum_{x=1}^{3} \sum_{y=1}^{3} (x-2)(y-2) = 2 \sum_{x=1}^{3} (x-2) \sum_{y=1}^{3} (y-2) = 0,$$

consequently,

$$E_{\min} = \frac{(6+6) - \sqrt{(6-6)^2 + 0^2}}{2} = 6$$

$$E_{\max} = \frac{(6+6) + \sqrt{(6-6)^2 + 0^2}}{2} = 6,$$

and so

$$\text{second-moment measure} = \frac{E_{\min}}{E_{\max}} = \frac{6}{6} = 1.$$

3

(2) Second-moment Measure for "Rectangle"

For "rectangle," we can compute that

$$a = \sum_{x=1}^{2}\sum_{y=1}^{4}(x-1.5)^2 = 4\sum_{x=1}^{2}(x-1.5)^2 = 2$$

$$c = \sum_{x=1}^{2}\sum_{y=1}^{4}(y-2.5)^2 = 2\sum_{y=1}^{4}(y-2.5)^2 = 10$$

$$b = 2\sum_{x=1}^{2}\sum_{y=1}^{4}(x-1.5)(y-2.5) = 2\sum_{x=1}^{2}(x-1.5)\sum_{y=1}^{4}(y-2.5) = 0,$$

consequently,

$$E_{\min} = \frac{(2+10)-\sqrt{(2-10)^2+0^2}}{2} = 2$$

$$E_{\max} = \frac{(2+10)+\sqrt{(2-10)^2+0^2}}{2} = 10,$$

and so

$$\text{second-moment measure} = \frac{E_{\min}}{E_{\max}} = \frac{2}{10} = 0.2.$$

(3) Second-moment Measure for "Diamond"

For "diamond," we think of it as a 5×5 square modulo the four "corners." Each pixel in the four corners has $B_{x,y} = 0$, whereas each pixel in the diamond has $B_{x,y} = 1$.

$$a = \sum_{x=1}^{5}\sum_{y=1}^{5}(x-3)^2 \cdot B_{xy}$$

$$= (1-3)^2 B_{1,3} + (2-3)^2 B_{2,2} + (2-3)^2 B_{2,3} + (2-3)^2 B_{2,4}$$

$$\quad + (4-3)^2 B_{4,2} + (4-3)^2 B_{4,3} + (4-3)^2 B_{4,4} + (5-3)^2 B_{5,3}$$

$$= 4+1+1+1+1+1+1+4 = 14$$

$$c = \sum_{x=1}^{5}\sum_{y=1}^{5}(y-3)^2 \cdot B_{xy} = 14 \quad \text{(by symmetry)}$$

$$b = 2\sum_{x=1}^{5}\sum_{y=1}^{5}(x-3)(y-3) \cdot B_{xy}$$

$$= 2[(2-3)(2-3) + (2-3)(4-3) + (4-3)(2-3) + (4-3)(4-3)] = 0,$$

consequently,

$$E_{\min} = \frac{(14+14)-\sqrt{(14-14)^2+0^2}}{2} = 14$$

$$E_{\max} = \frac{(14+14)+\sqrt{(14-14)^2+0^2}}{2} = 14,$$

4

and so

$$\text{second-moment measure} = \frac{E_{\min}}{E_{\max}} = \frac{14}{14} = 1.$$

(4) Second-moment Measure for "Stick"

For "rectangle," we can compute that

$$a = \sum_{y=1}^{10}(0.5 - 0.5)^2 = 0$$

$$c = \sum_{x=1}^{1}\sum_{y=1}^{10}(y - 5.5)^2 = \sum_{y=1}^{10}(y - 2.5)^2 = 82.5$$

$$b = 2\sum_{x=1}^{1}\sum_{y=1}^{10}(x - 0.5)(y - 5.5) = \sum_{y=1}^{10}(y - 5.5) = 0,$$

consequently,

$$E_{\min} = \frac{(0 + 82.5) - \sqrt{(0 - 82.5)^2 + 0^2}}{2} = 0$$

$$E_{\max} = \frac{(0 + 82.5) + \sqrt{(0 - 82.5)^2 + 0^2}}{2} = 82.5,$$

and so

$$\text{second-moment measure} = \frac{E_{\min}}{E_{\max}} = \frac{0}{82.5} = 0.$$

Part 3. Compute μ/σ for each shape

For this part, we assume the coordinate for each centroid is $(0,0)$.

(1) μ/σ for "Square"

$$\mu = \frac{4\sqrt{2} + 4}{8} = \frac{\sqrt{2} + 1}{2} \approx 1.207$$

$$\sigma = \sqrt{\frac{4(\sqrt{2})^2 + 4}{8} - \mu^2} = \sqrt{\frac{3 - 2\sqrt{2}}{4}} \approx 0.2071,$$

consequently,

$$\frac{\mu}{\sigma} \approx \frac{1.207}{0.2071} \approx 5.8281.$$

(2) μ/σ for "Rectangle"

$$\mu = \frac{4\sqrt{2.5} + 4\sqrt{0.5}}{8} = \frac{\sqrt{10} + \sqrt{2}}{4} \approx 1.1441$$

$$\sigma = \sqrt{\frac{4 \times 2.5 + 4 \times 0.5}{8} - \mu^2} = \frac{\sqrt{3 - \sqrt{5}}}{2} \approx 0.4370,$$

consequently,

$$\frac{\mu}{\sigma} \approx \frac{1.1441}{0.4370} \approx 2.6181.$$

(3) μ/σ for "Diamond"

$$\mu = \frac{4\sqrt{2} + 4 \times 2}{8} = \frac{2 + \sqrt{2}}{2} \approx 1.7071$$

$$\sigma = \sqrt{\frac{4 \times 2 + 4 \times 4}{8} - \mu^2} = \frac{\sqrt{6 - 4\sqrt{2}}}{2} = \frac{2 - \sqrt{2}}{2} \approx 0.2929,$$

consequently,

$$\frac{\mu}{\sigma} \approx \frac{1.7071}{0.2929} \approx 5.8283.$$

(4) μ/σ for "Stick"

$$\mu = \frac{2 \times (4.5 + 3.5 + 2.5 + 1.5 + 0.5)}{10} = \frac{25}{10} = 2.5$$

$$\sigma = \sqrt{\frac{2 \times (4.5^2 + 3.5^2 + 2.5^2 + 1.5^2 + 0.5^2)}{10} - \mu^2} = \sqrt{2} \approx 1.4142,$$

consequently,

$$\frac{\mu}{\sigma} \approx \frac{2.5}{1.4142} \approx 1.7678.$$

Exercise 3: Least Squares Method

First, let's impose coordinate values onto the red squares. Here's the list of all such coordinate values, each of which corresponds to a red square in the figure from left to right: $(0, 5)$, $(1, 3)$, $(2, 8)$, $(3, 6)$, $(4, 1)$, $(5, 7)$, $(6, 4)$, $(7, 9)$, $(8, 3)$, $(9, 6)$.

Then, noting that we'll use L_2-distance, and denoting the sum of distance as $d(x, y)$ where (x, y) is the coordinate of the unknown "optimized" pixel in the figure, we have

$$\begin{aligned} d(x, y) = &\sqrt{(0 - x)^2 + (5 - y)^2} + \sqrt{(1 - x)^2 + (3 - y)^2} + \sqrt{(2 - x)^2 + (8 - y)^2} \\ &+ \sqrt{(3 - x)^2 + (6 - y)^2} + \sqrt{(4 - x)^2 + (1 - y)^2} + \sqrt{(5 - x)^2 + (7 - y)^2} \\ &+ \sqrt{(6 - x)^2 + (4 - y)^2} + \sqrt{(7 - x)^2 + (9 - y)^2} + \sqrt{(8 - x)^2 + (3 - y)^2} \\ &+ \sqrt{(9 - x)^2 + (6 - y)^2}. \end{aligned}$$

We want to minimize $d(x, y)$, so we take the partial derivatives of $d(x, y)$ with respect to x and y, respectively, like so:

$$\frac{\partial}{\partial x}d(x,y) = -\frac{x}{\sqrt{(0-x)^2+(5-y)^2}} - \frac{1-x}{\sqrt{(1-x)^2+(3-y)^2}} - \frac{2-x}{\sqrt{(2-x)^2+(8-y)^2}}$$
$$- \frac{3-x}{\sqrt{(3-x)^2+(6-y)^2}} - \frac{4-x}{\sqrt{(4-x)^2+(1-y)^2}} - \frac{5-x}{\sqrt{(5-x)^2+(7-y)^2}}$$
$$- \frac{6-x}{\sqrt{(6-x)^2+(4-y)^2}} - \frac{7-x}{\sqrt{(7-x)^2+(9-y)^2}} - \frac{8-x}{\sqrt{(8-x)^2+(3-y)^2}}$$
$$- \frac{9-x}{\sqrt{(9-x)^2+(6-y)^2}}$$
$$= 0$$

$$\frac{\partial}{\partial y}d(x,y) = -\frac{5-y}{\sqrt{(0-x)^2+(5-y)^2}} - \frac{3-y}{\sqrt{(1-x)^2+(3-y)^2}} - \frac{8-y}{\sqrt{(2-x)^2+(8-y)^2}}$$
$$- \frac{6-y}{\sqrt{(3-x)^2+(6-y)^2}} - \frac{1-y}{\sqrt{(4-x)^2+(1-y)^2}} - \frac{7-y}{\sqrt{(5-x)^2+(7-y)^2}}$$
$$- \frac{4-y}{\sqrt{(6-x)^2+(4-y)^2}} - \frac{9-y}{\sqrt{(7-x)^2+(9-y)^2}} - \frac{3-y}{\sqrt{(8-x)^2+(3-y)^2}}$$
$$- \frac{6-y}{\sqrt{(9-x)^2+(6-y)^2}}$$
$$= 0$$

I do not recall how to minimize the sum of square roots. But since this is a convex expression, I think it reasonable to say that the rest of the task can be accomplished via convex optimization.